

# Expected Utility and Post-Retirement Investment and Spending Strategies

William F. Sharpe

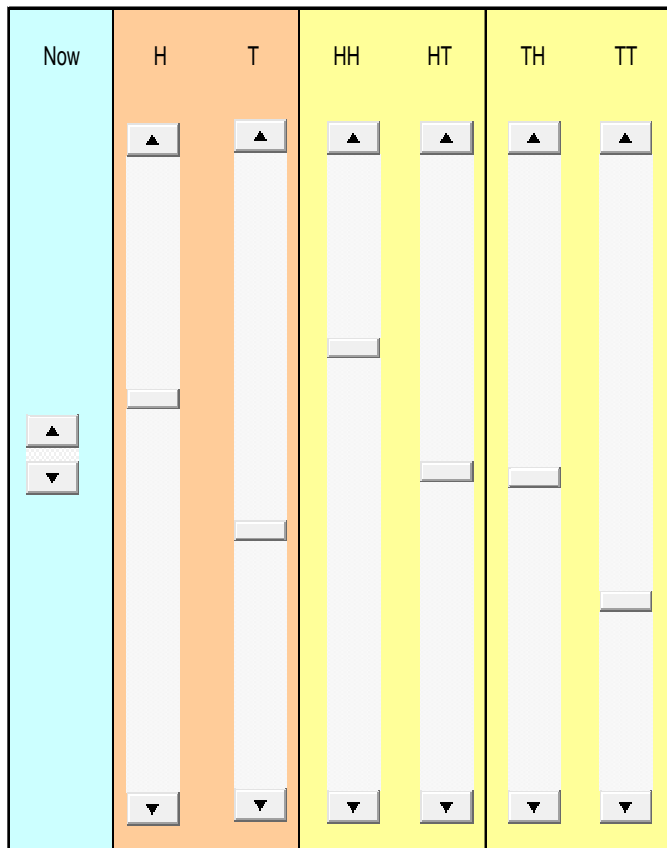
STANCO 25 Professor of Finance  
Stanford University

[www.wsharpe.com](http://www.wsharpe.com) William F. Sharpe

# Choosing a Post-retirement Financial Plan

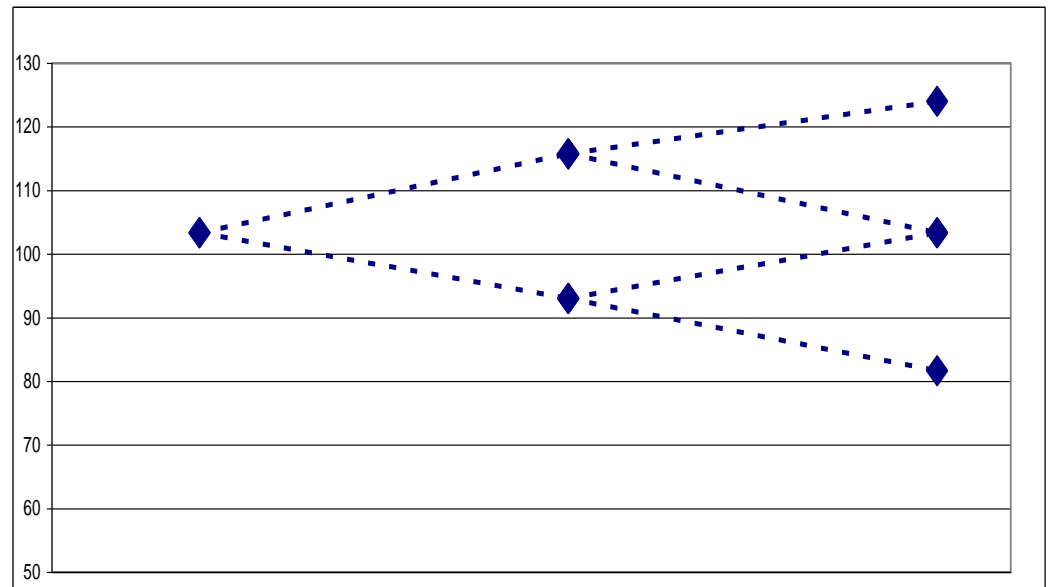
Wealth

Prices

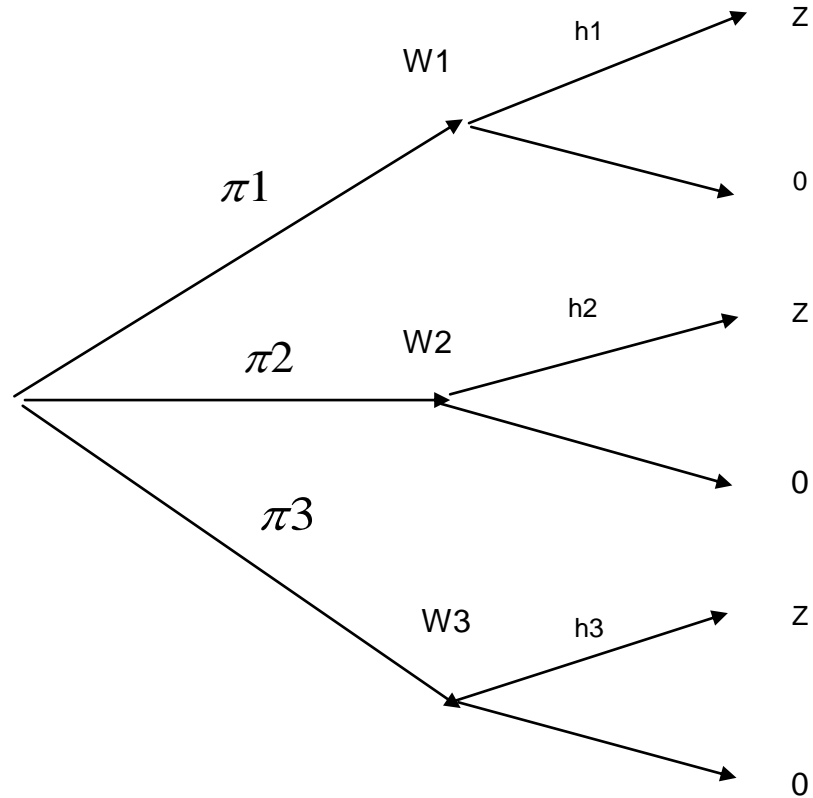


| Spent | Spent | Spent | Spent | Spent | Spent | Spent |
|-------|-------|-------|-------|-------|-------|-------|
| 1.00  | 1.12  | 0.90  | 1.20  | 1.00  | 1.00  | 0.79  |

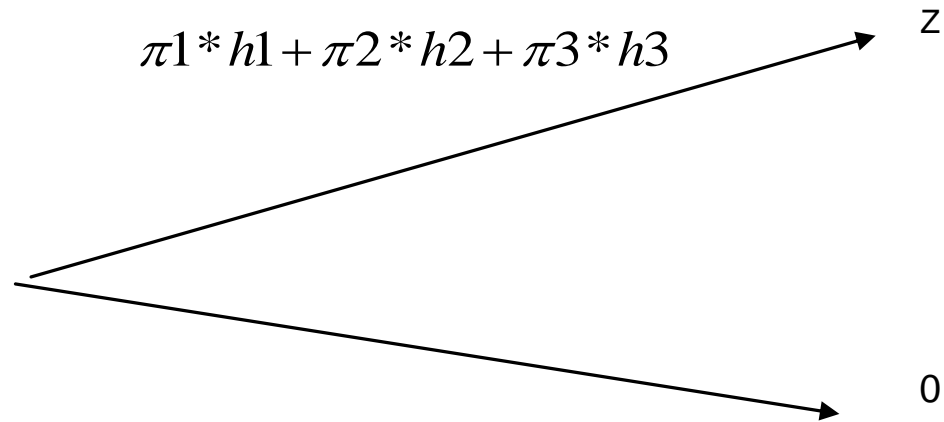
|    | Year 1   | Year 2   | Year 3   |
|----|----------|----------|----------|
| HH | \$103.37 | \$115.77 | \$124.04 |
| HT | \$103.37 | \$115.77 | \$103.37 |
| TH | \$103.37 | \$93.03  | \$103.37 |
| TT | \$103.37 | \$93.03  | \$81.66  |



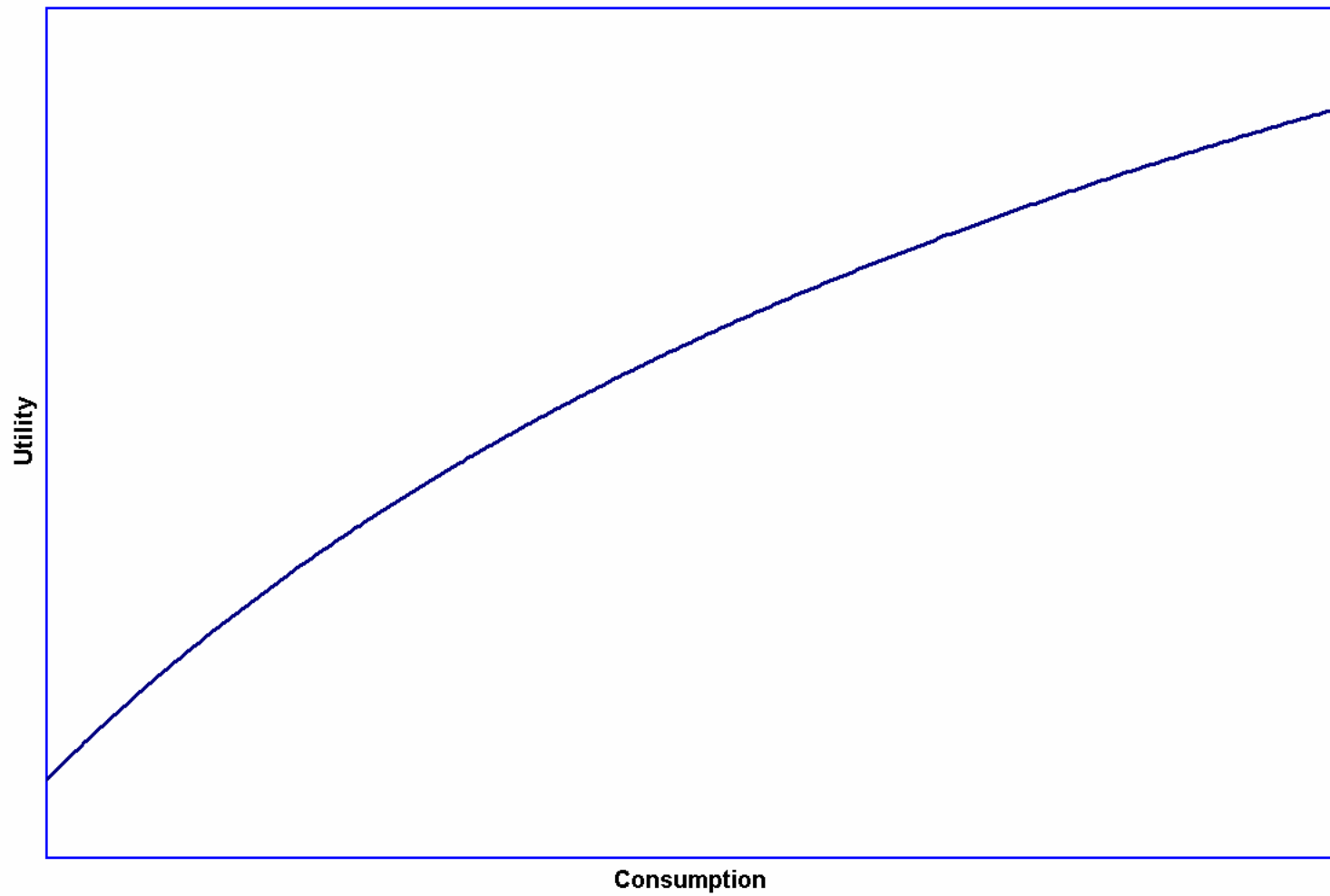
# Von Neumann-Morgenstern (1)



# Von Neumann-Morgenstern (2)



# Utility



# Expected Utility

$$EU = f(X_1, \dots, X_n; \pi_1, \dots, \pi_n)$$

$$EU = \sum \pi_s u_s(X_s)$$

# First-order Conditions for Maximizing Expected Utility

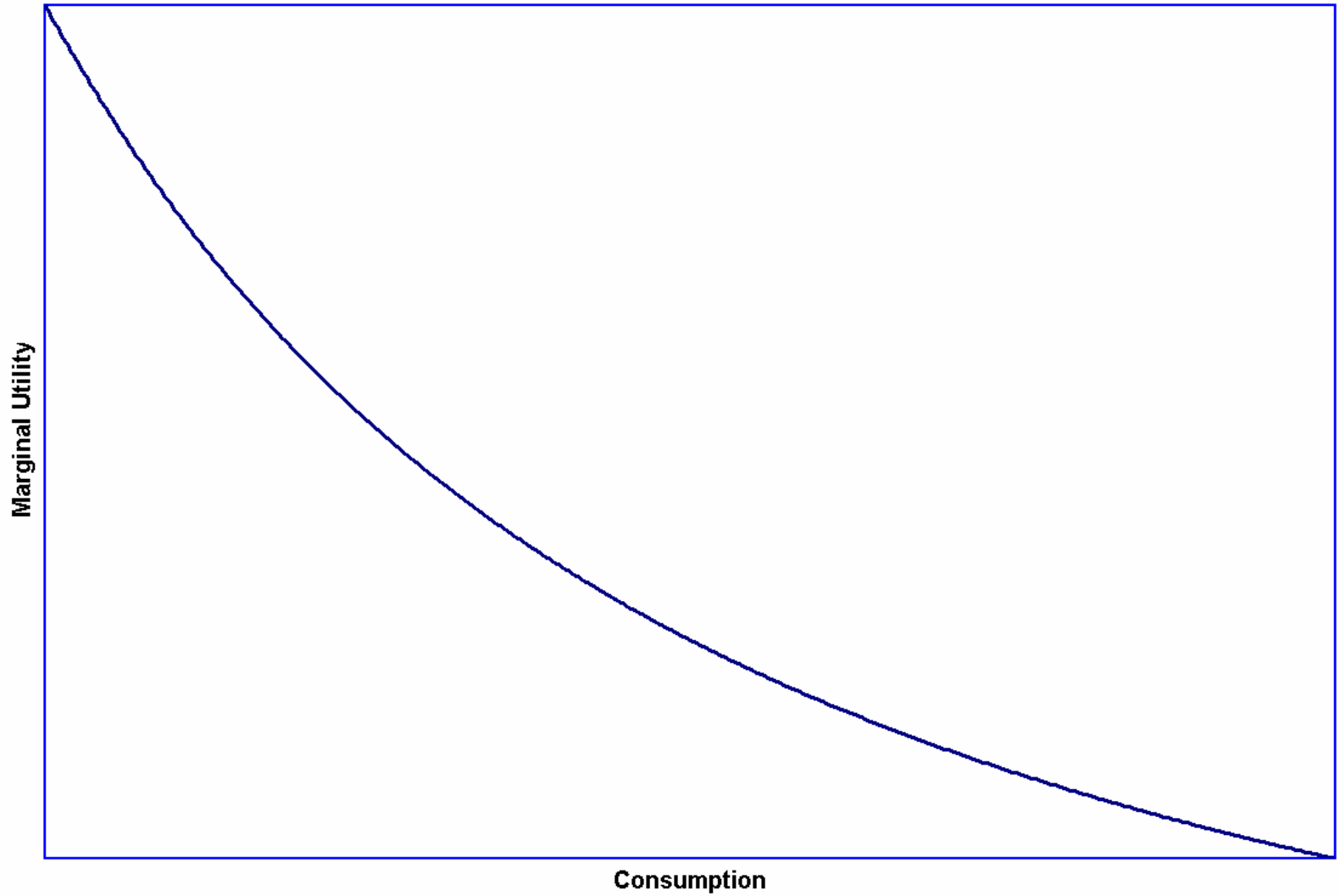
$$\text{Max} \sum \pi_s u_s (X_s)$$

$$\text{s.t.} \sum p_s X_s = W$$

$$\pi_s u' (X_s) = \lambda p_s$$

$$m u_s (X_s) = \lambda P P C_s$$

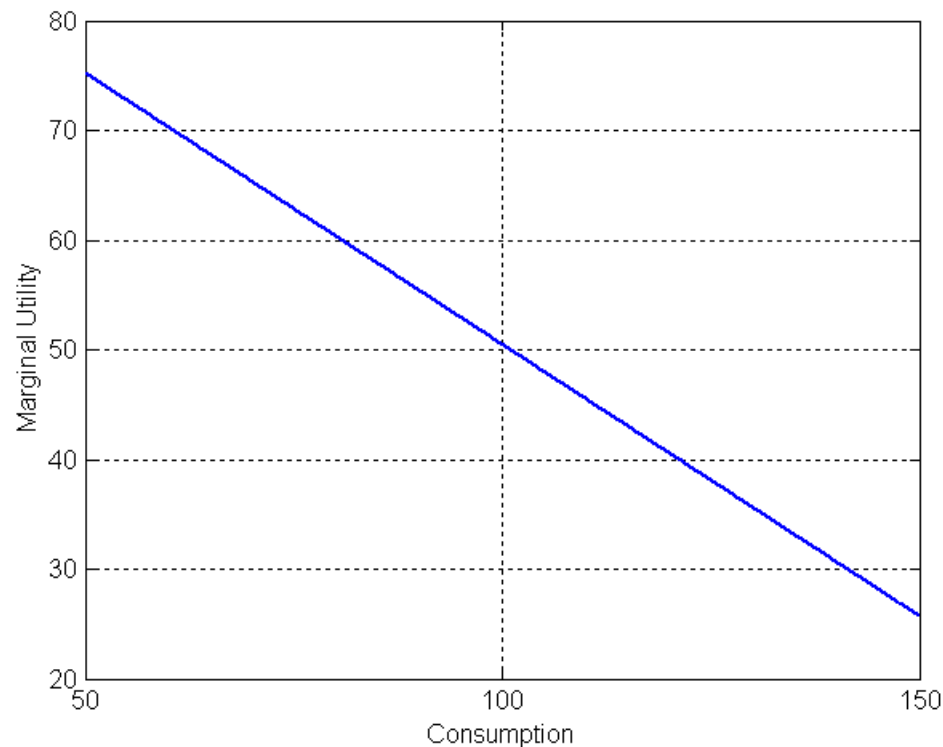
# Marginal Utility



# Single-period Utility functions

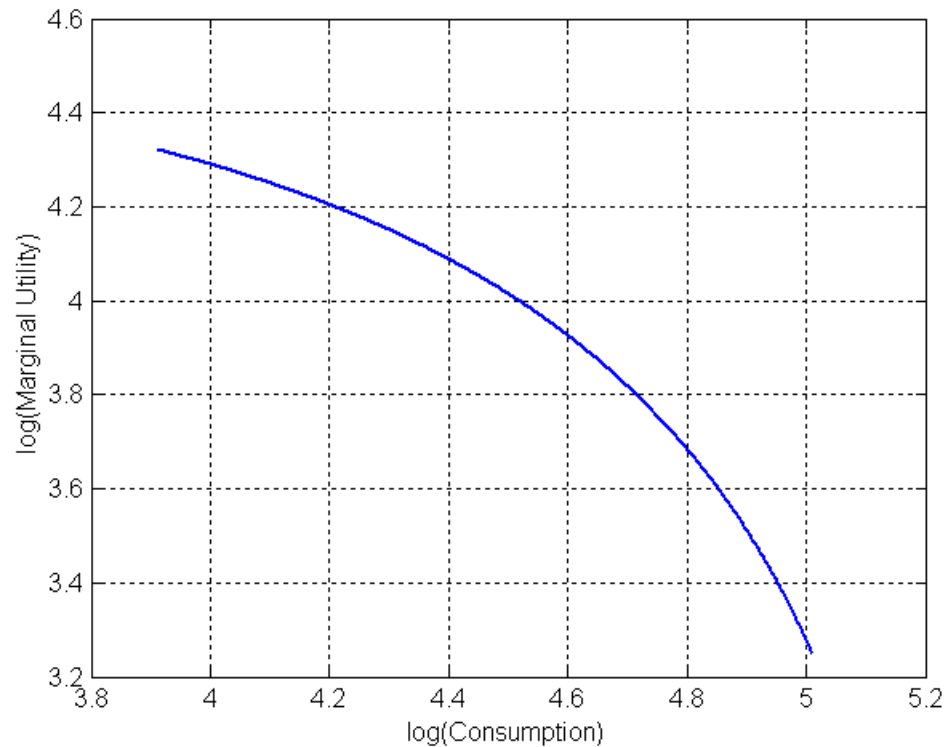
- Quadratic (Mean/Variance)
- Constant Relative Risk Aversion (CRRA)
- Hyperbolic Absolute Risk Aversion (HARA)
- Prospect theory

# A Quadratic Utility Marginal Utility Function

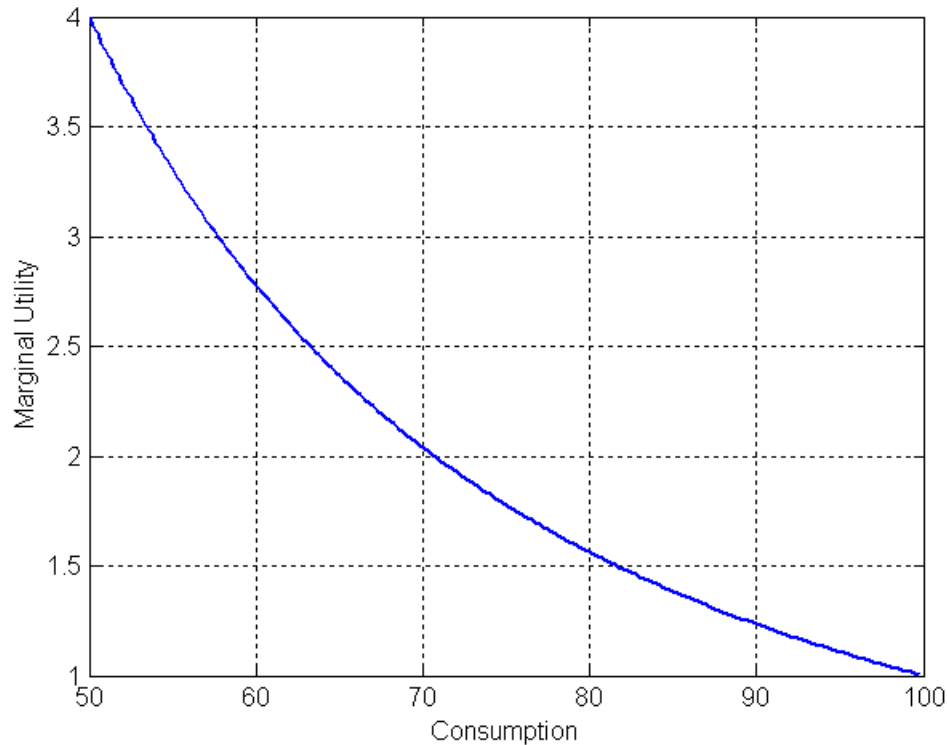


$$m = b - cX$$

# A Quadratic Utility Marginal Utility Function (log/log scale)

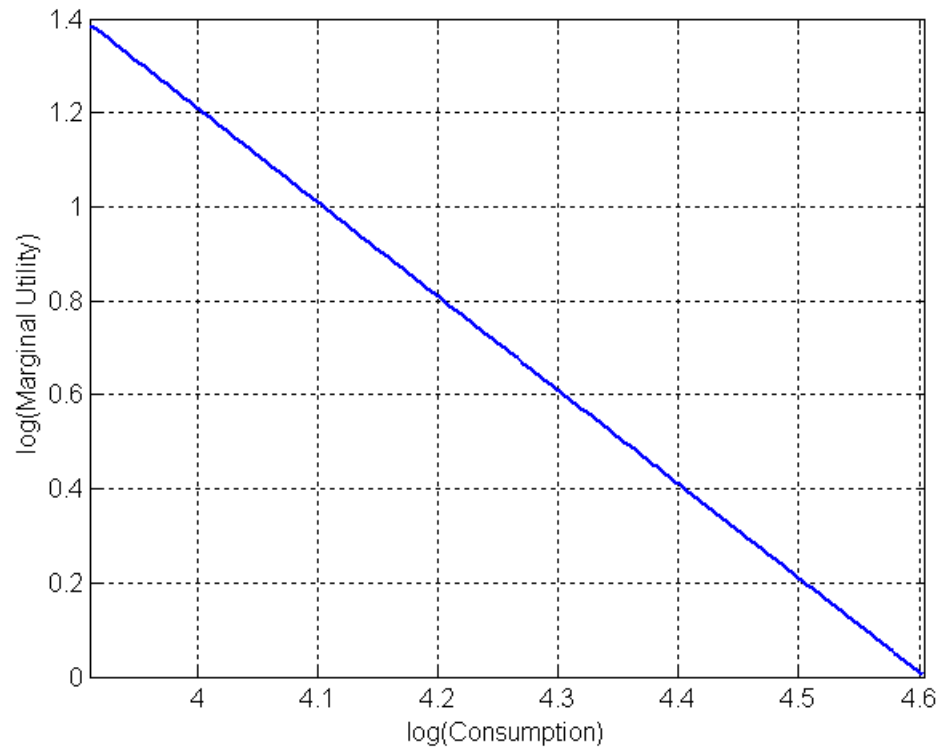


# A CRRA Marginal Utility Function



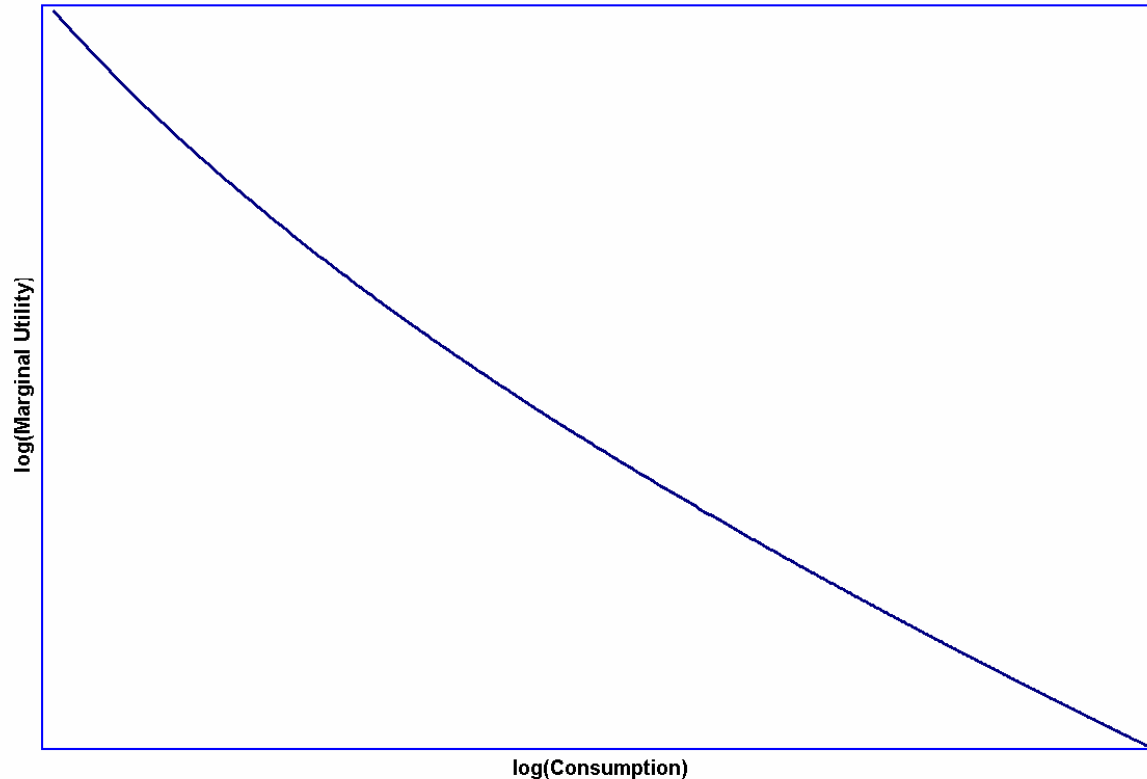
$$m = X^{-b}$$

# A CRRA Marginal Utility Function: (log/log scale)



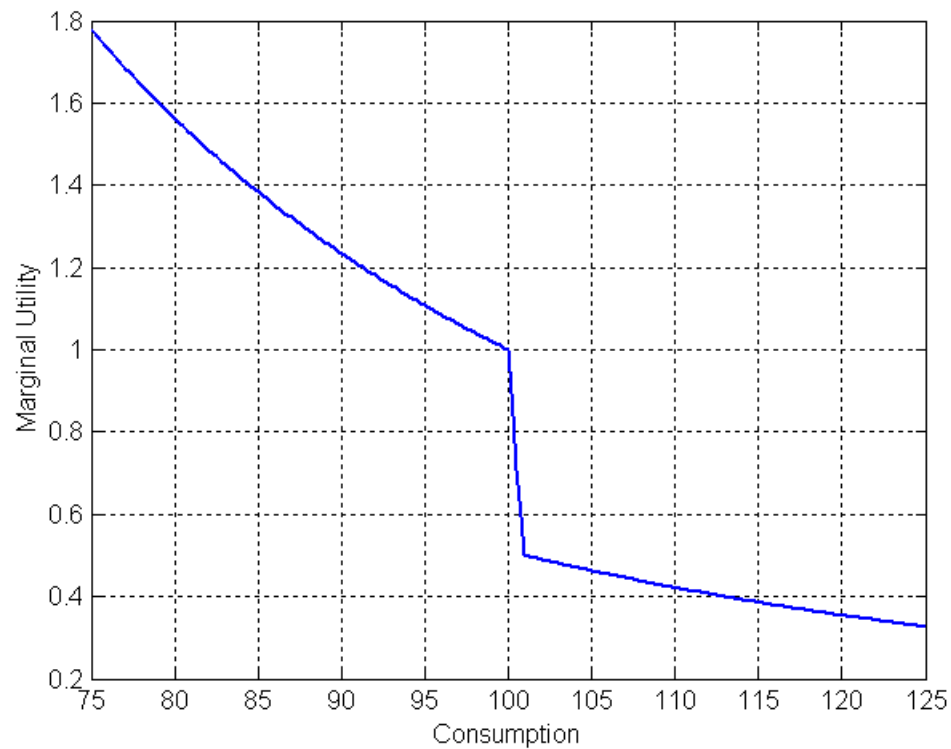
$$\ln m = -b \ln X$$

# A HARA Marginal Utility Function (log/log scale)

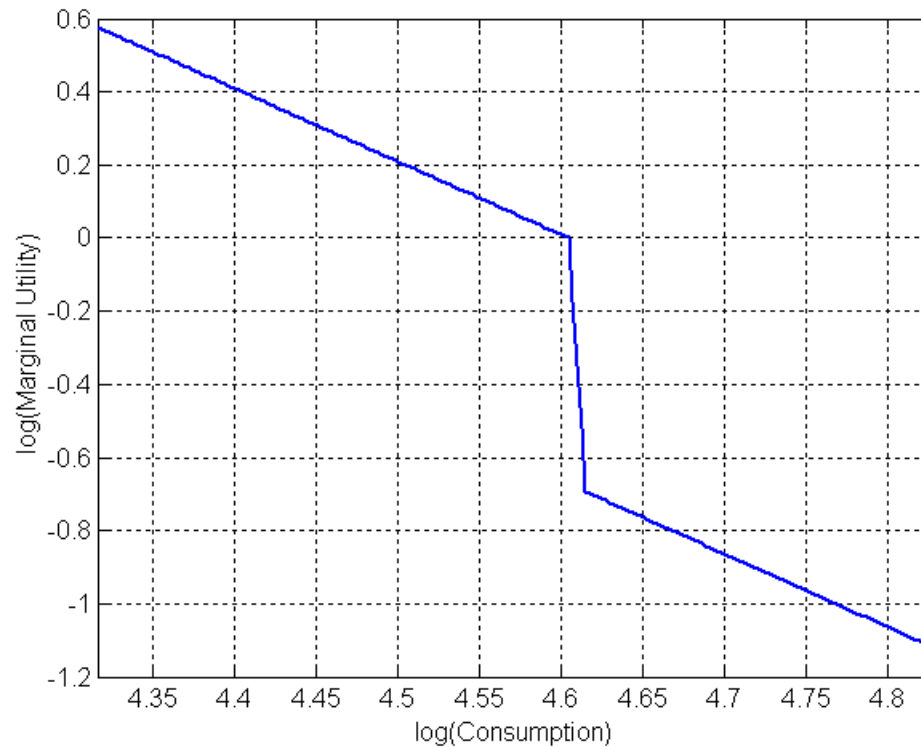


$$\ln m = -b \ln(X - M)$$

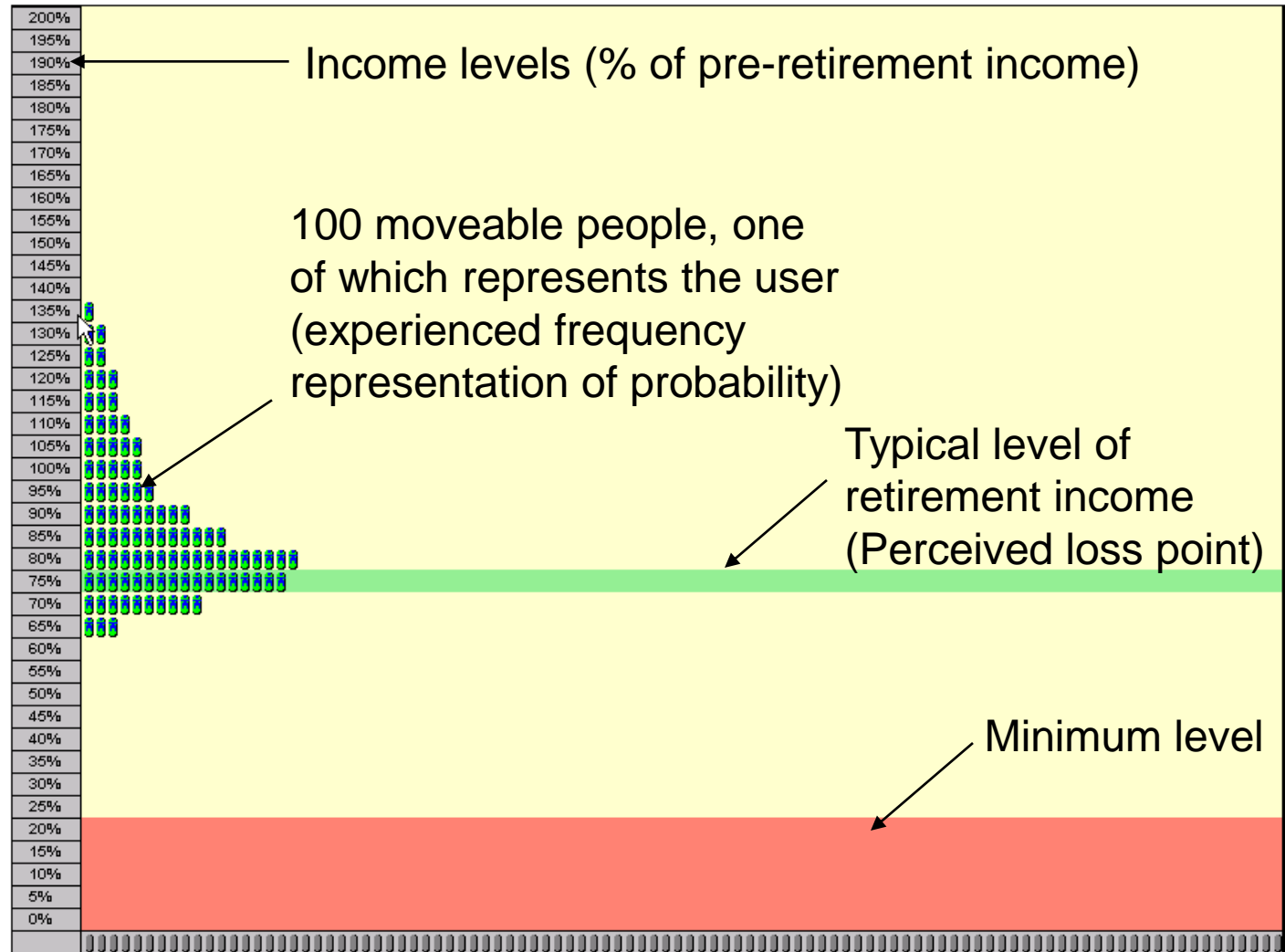
# A Kinked Marginal Utility Function



# A Kinked Marginal Utility Function (log/log scale)



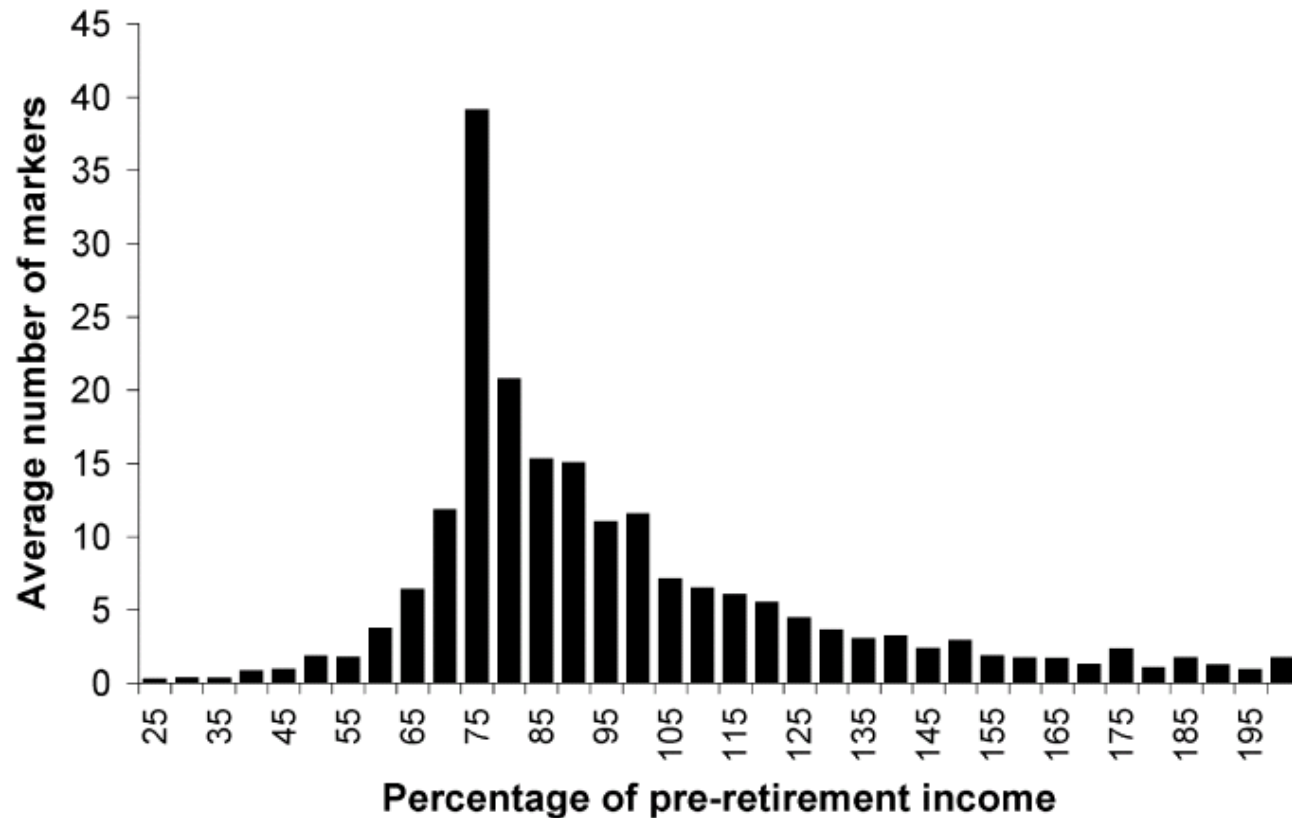
# The Distribution Builder



# Average Choices

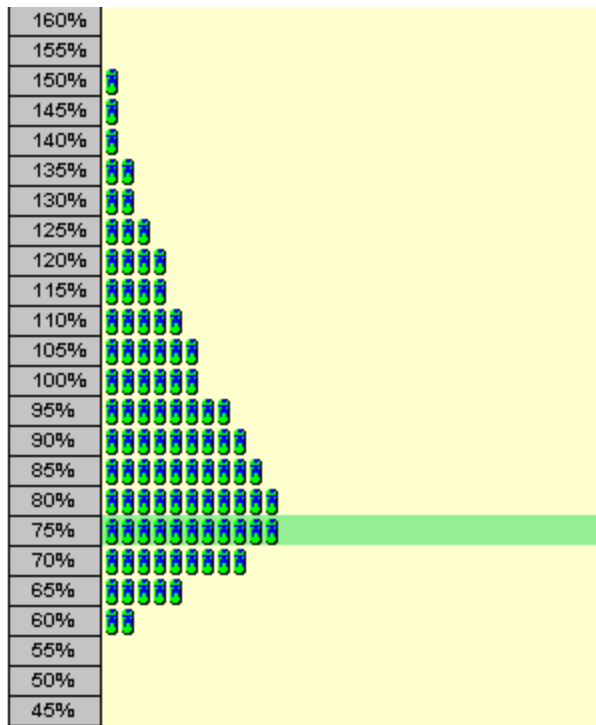
FIGURE 3

AGGREGATE DISTRIBUTION

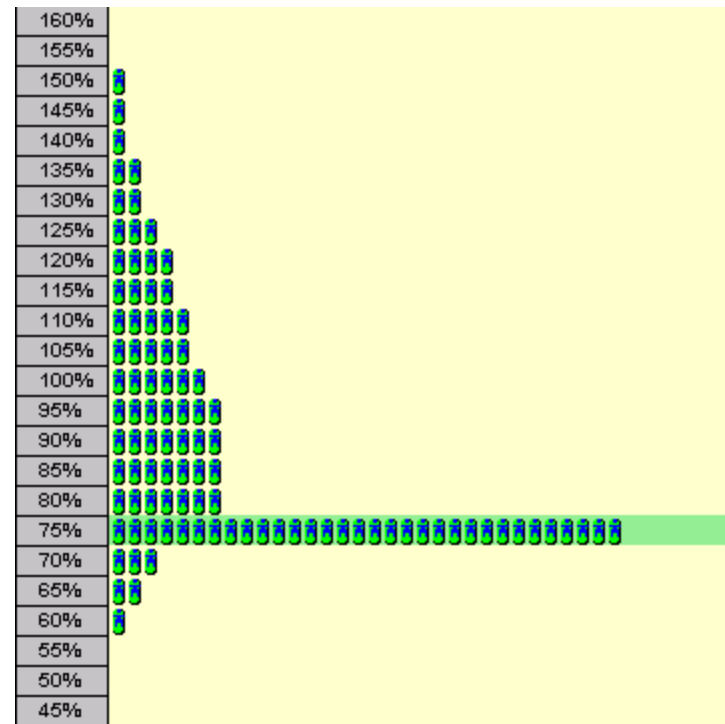


# Types of Choices

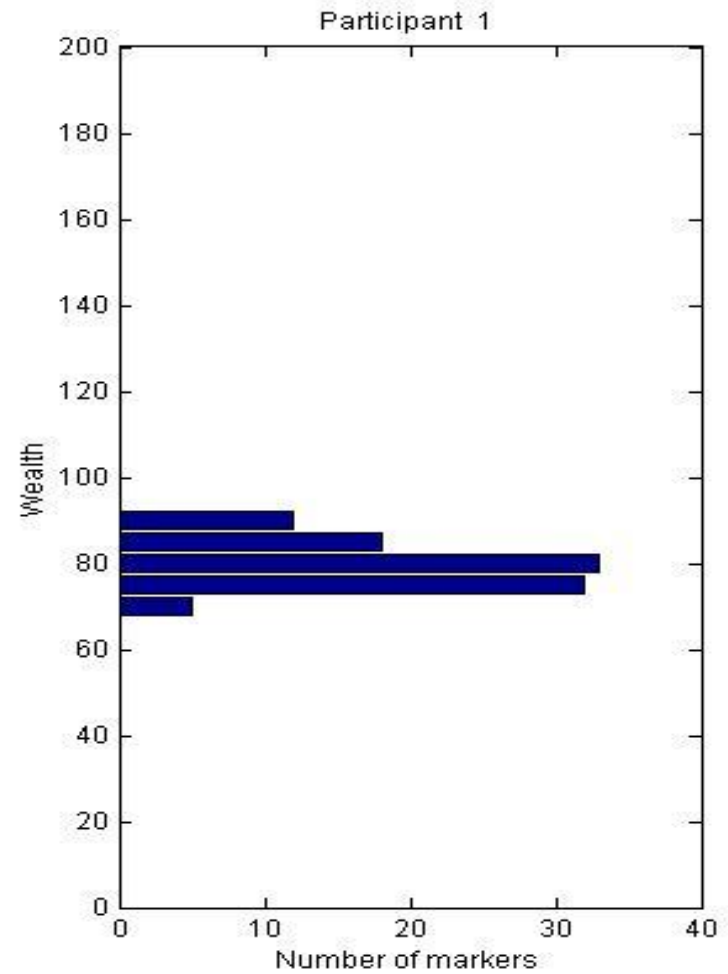
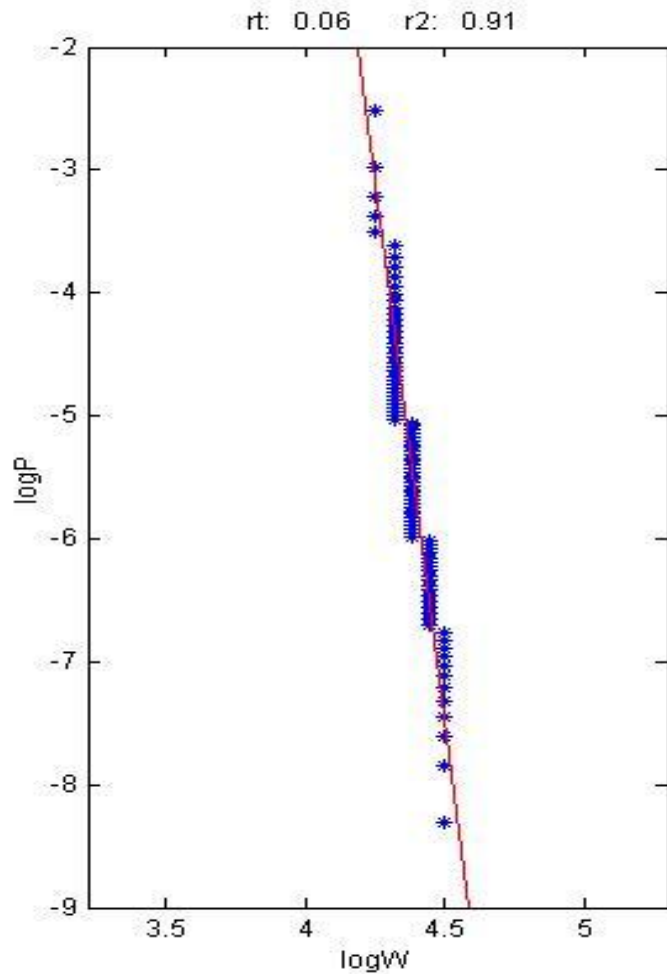
Do preferences conform with maximization of a CRRA utility function?



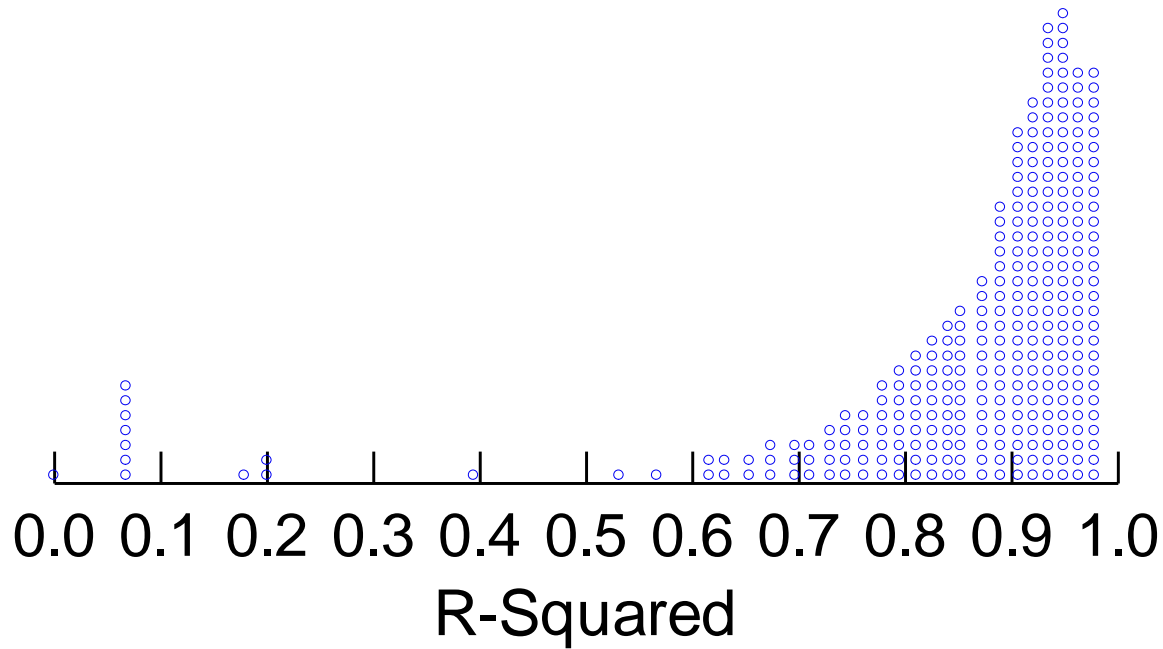
Or do preferences exhibit loss aversion?



# Testing for CRRA Preferences



# Distribution of R-squared Values for CRRA Utility

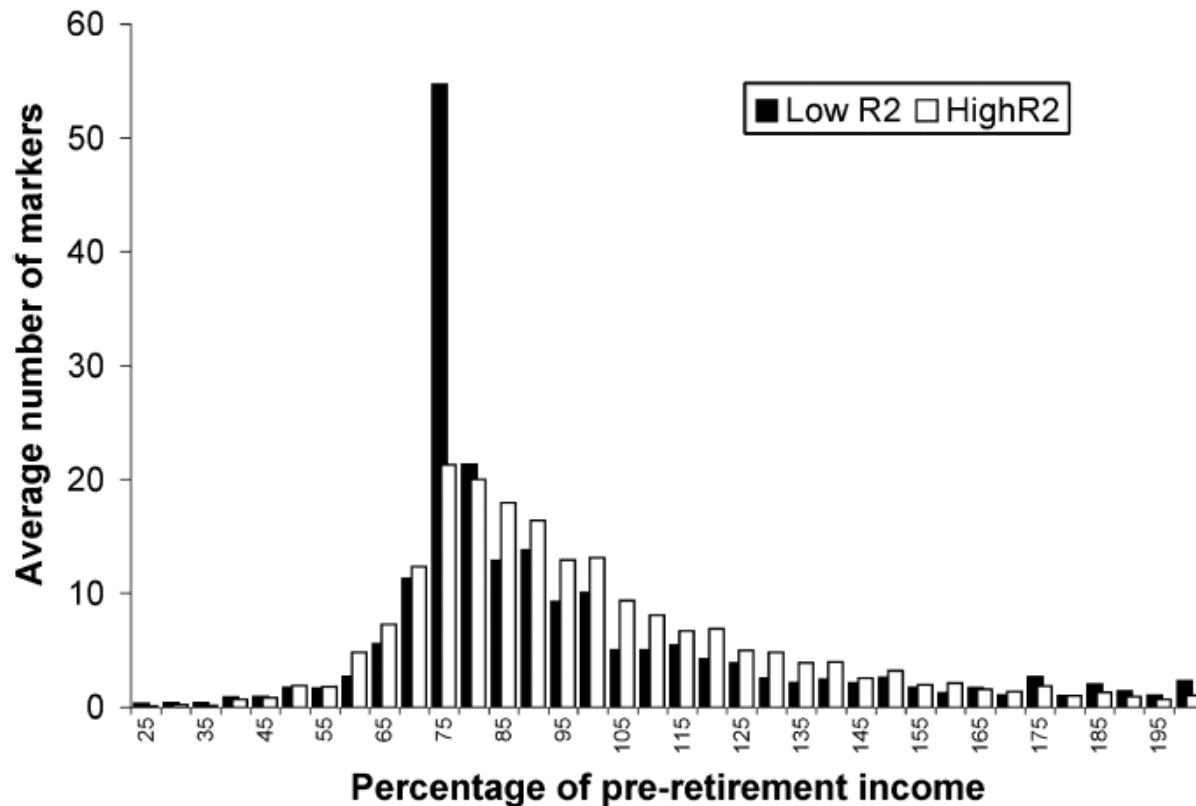


# Distribution Builder Results

## Split at $R^2=0.90$ (approx. median)

FIGURE 5

POPULATION SPLIT BY FIT TO CRRA MODEL



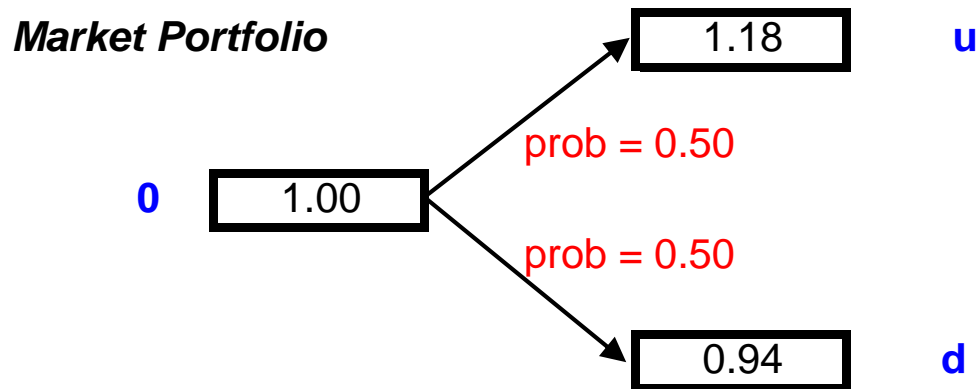
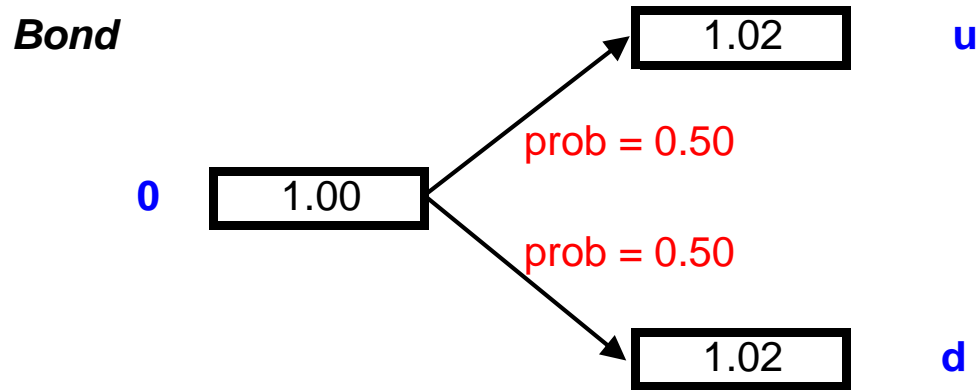
# Multi-period Financial Plans

- Multiple time periods
- For each time period, multiple possible states of the world
  - mutually exclusive
  - exhaustive
- Objective:
  - Select consumption for each time and state to maximize expected utility, subject to a budget constraint

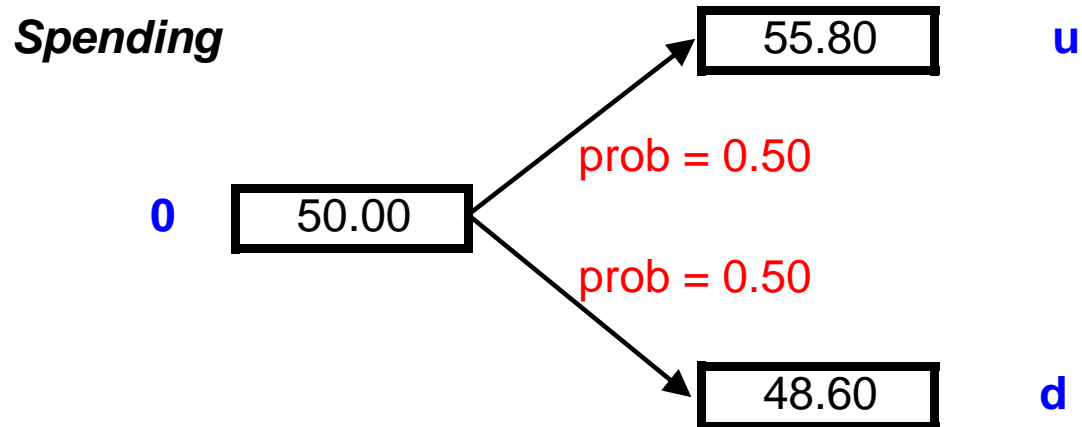
# The Simplest Possible Risky Capital Market

- Two periods
  - Now
  - Next year
- Two future *states of the world*
  - The market is up
  - The market is down
- Two securities
  - A riskless real bond
  - A portfolio of risky securities in market proportions

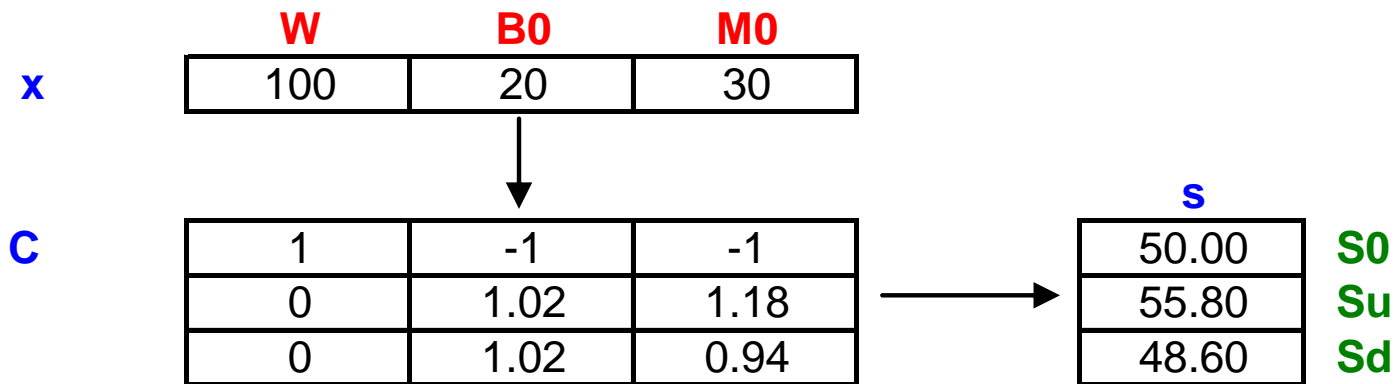
# Capital Market Characteristics



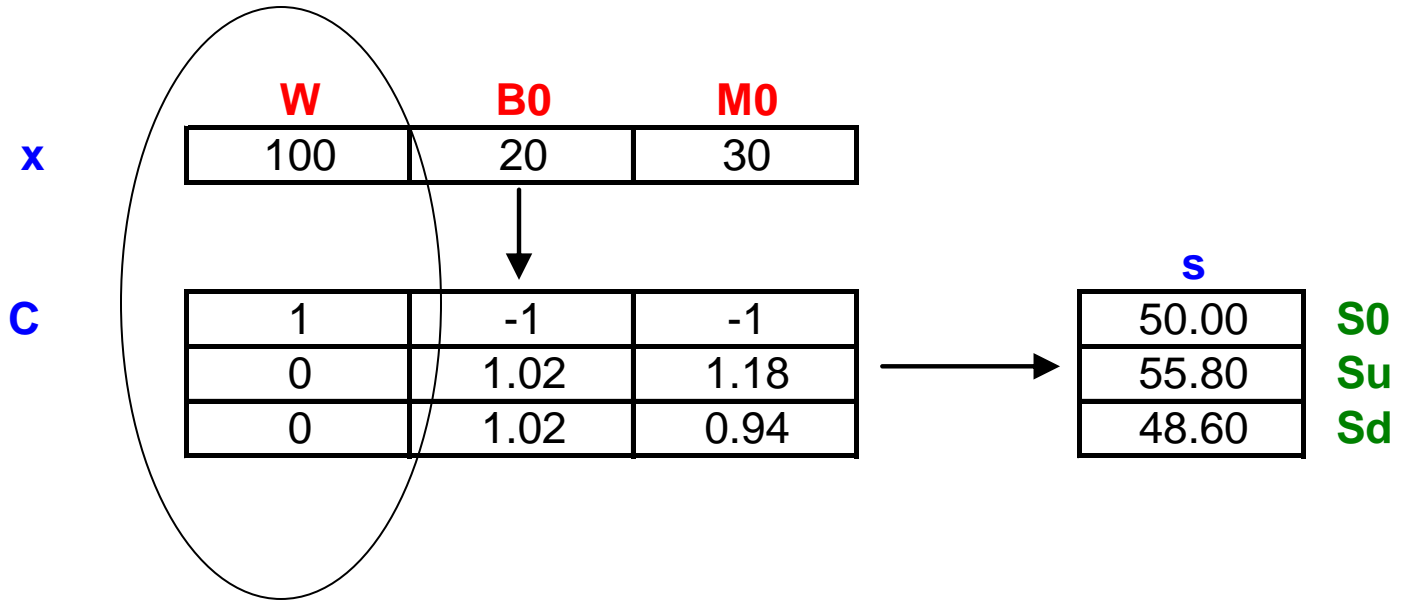
# Desired Spending



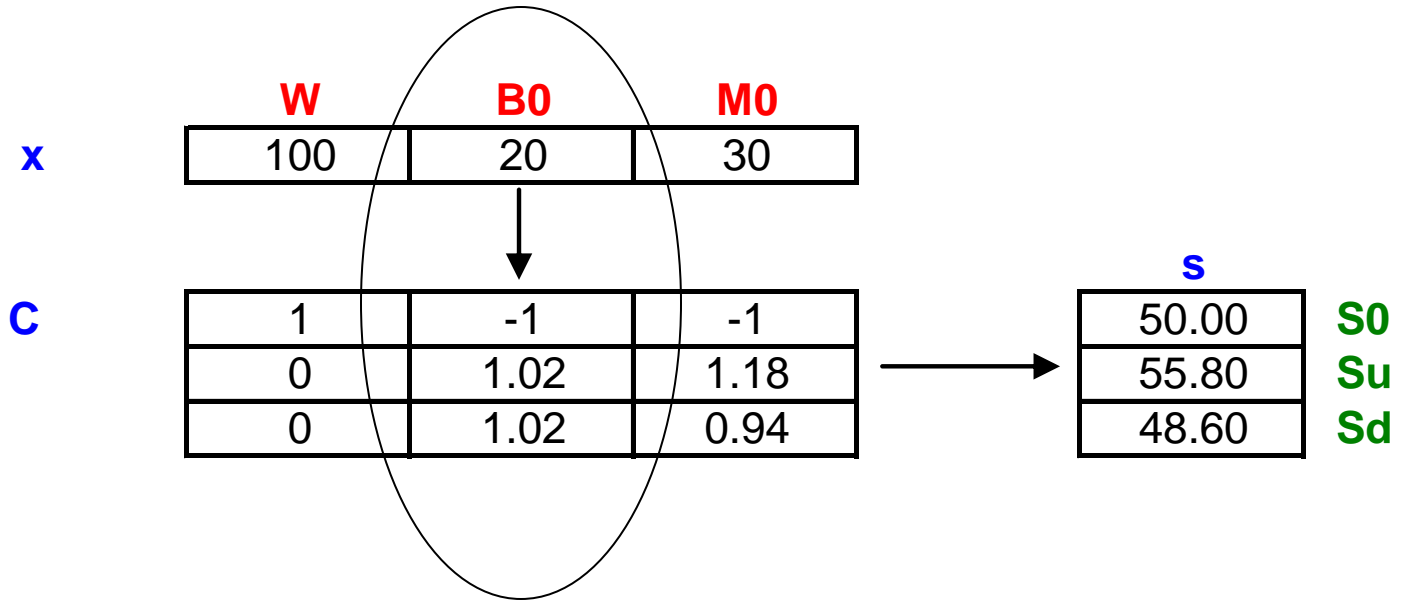
# Wealth, Financial Strategy and Desired Spending



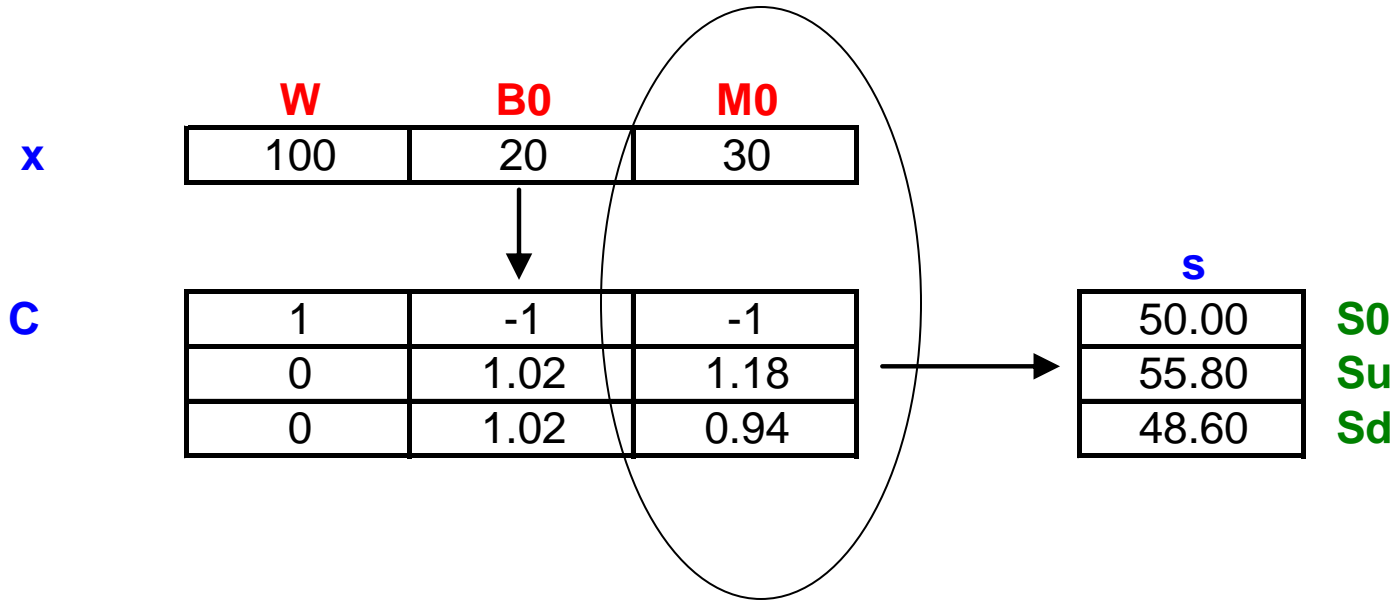
# Initial Wealth



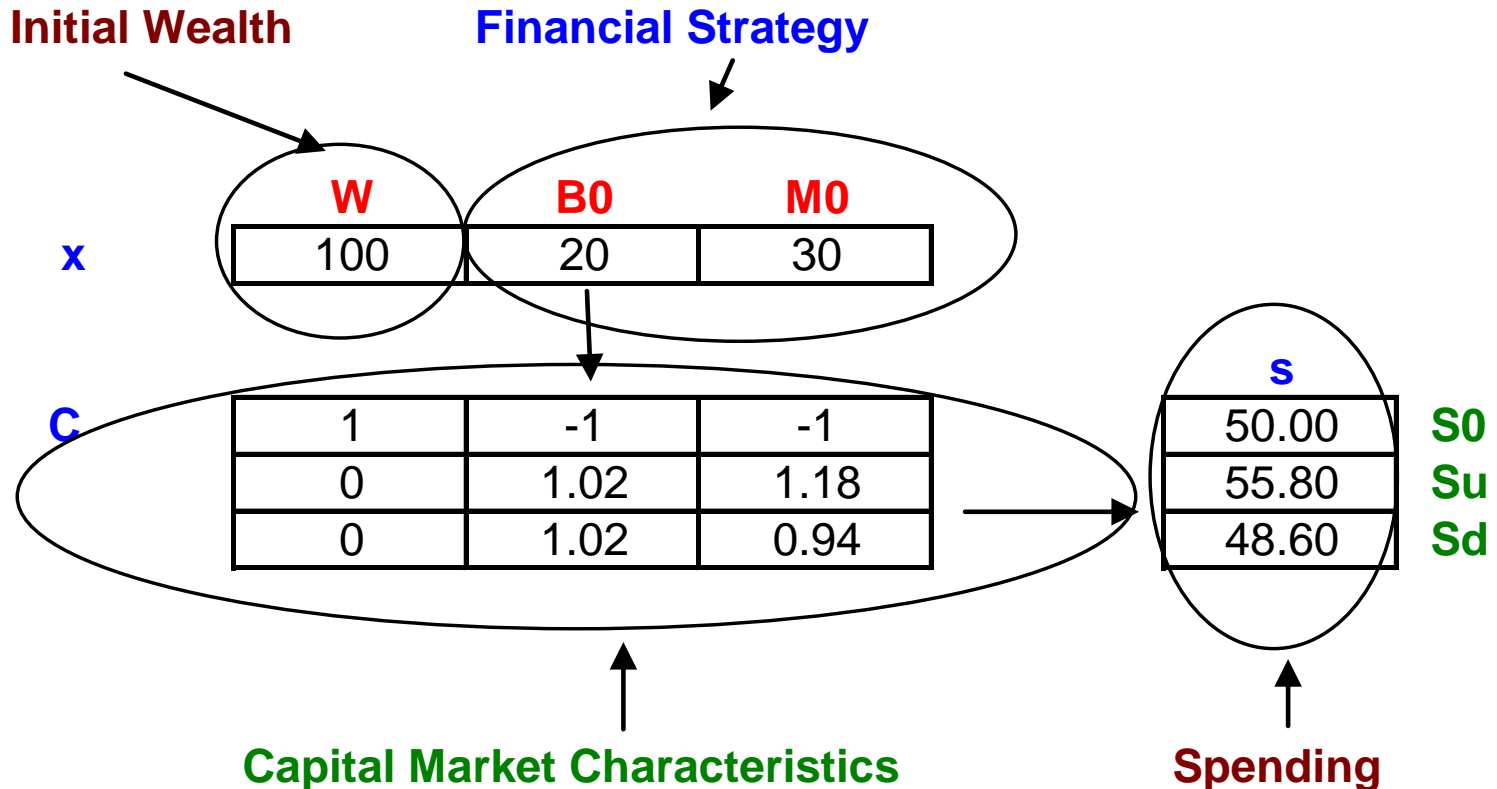
# Bond Investment



# Market Portfolio Investment

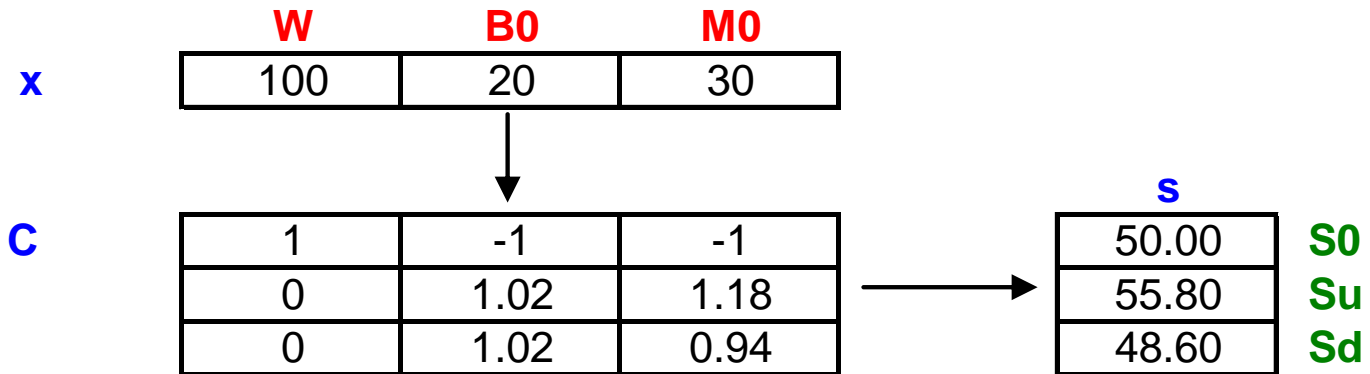


# Wealth, Financial Strategy, Capital Markets and Spending



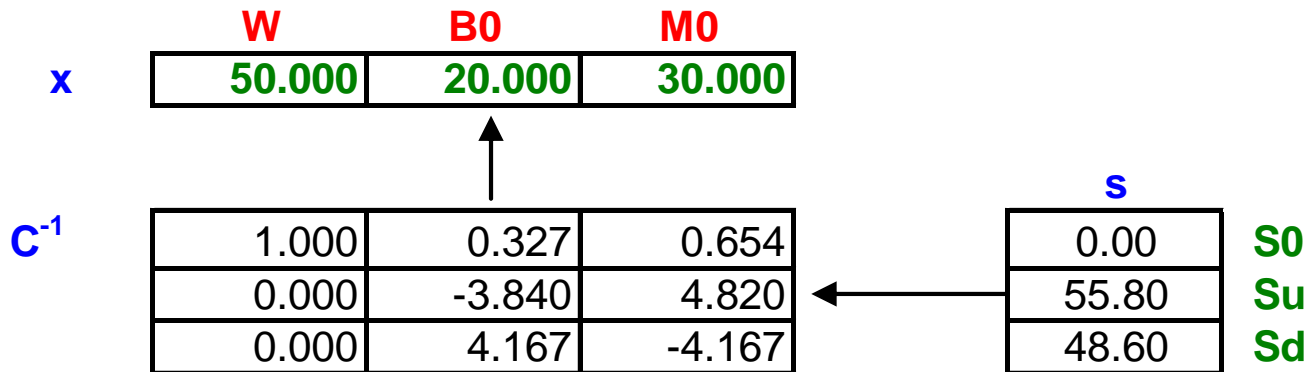
# Decisions → Spending

$$Cx' = s$$

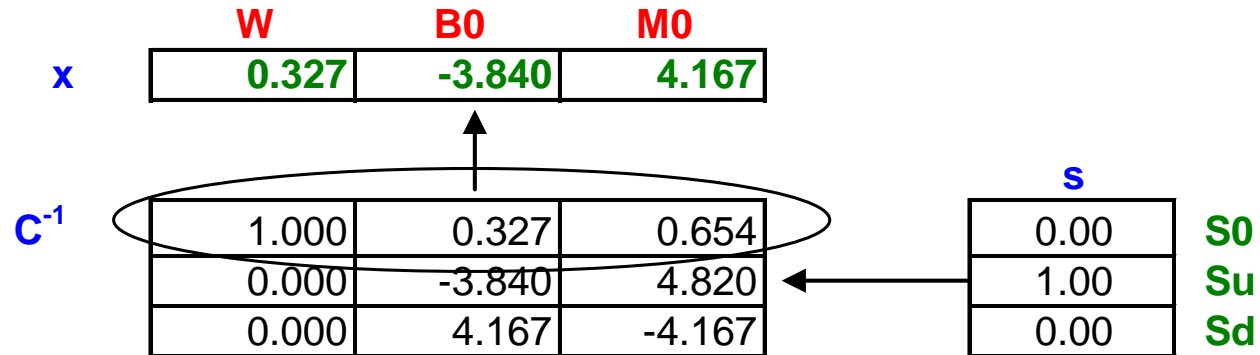


# Decisions ← Spending

$$x' = C^{-1}s$$



# Arrow-Debreu Prices



# Lockbox Strategies

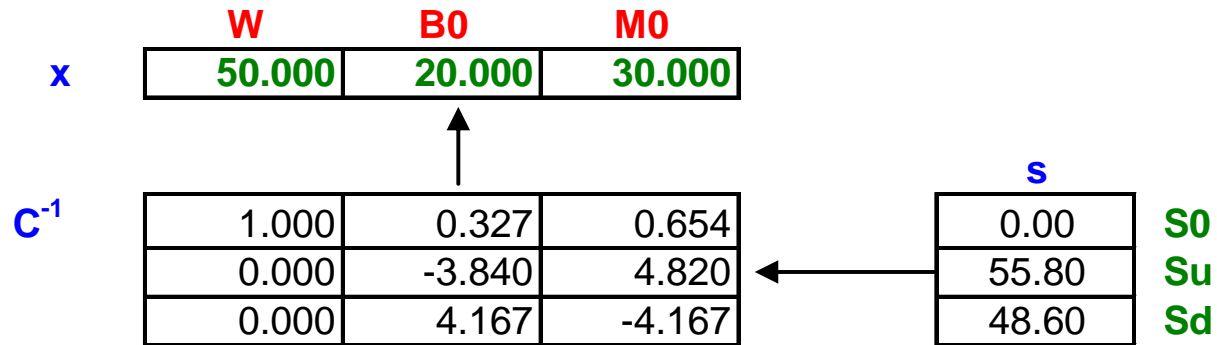
$$x' = C^{-1} s$$

$$x' = C^{-1} (s_1 + s_2)$$

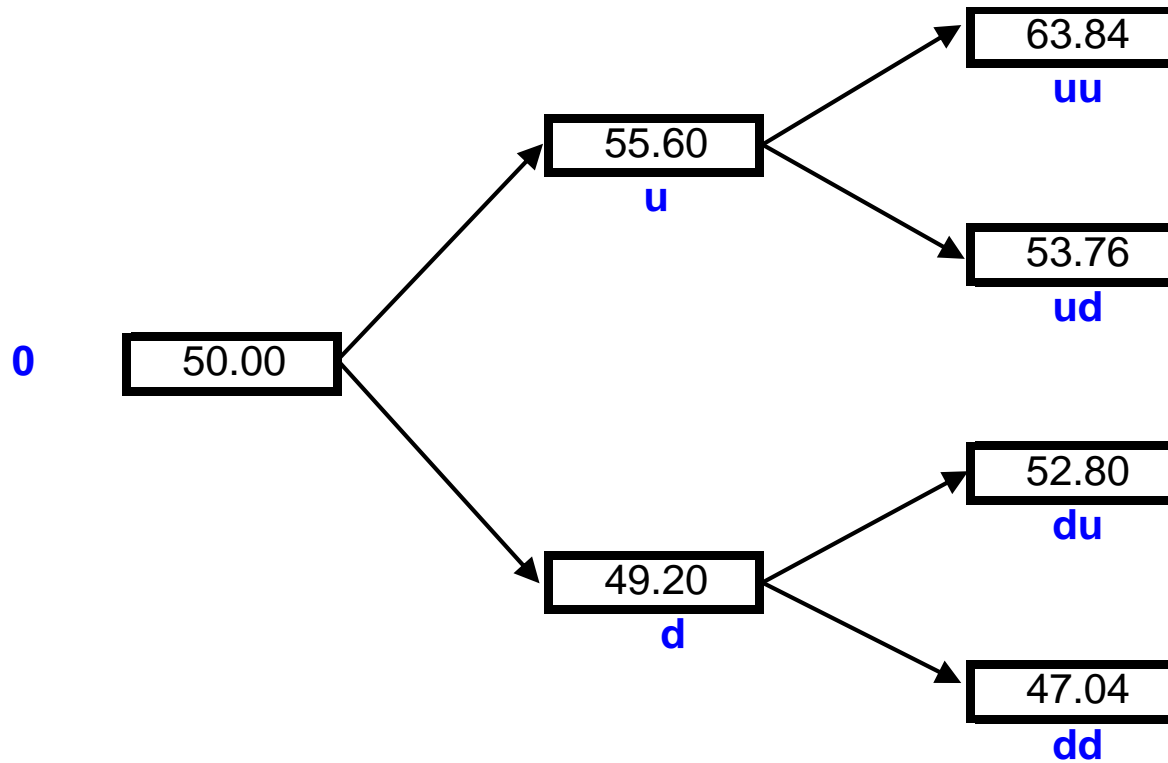
$$x' = C^{-1} s_1 + C^{-1} s_2$$

$$x' = x_1 + x_2$$

# Lockbox, Period 1



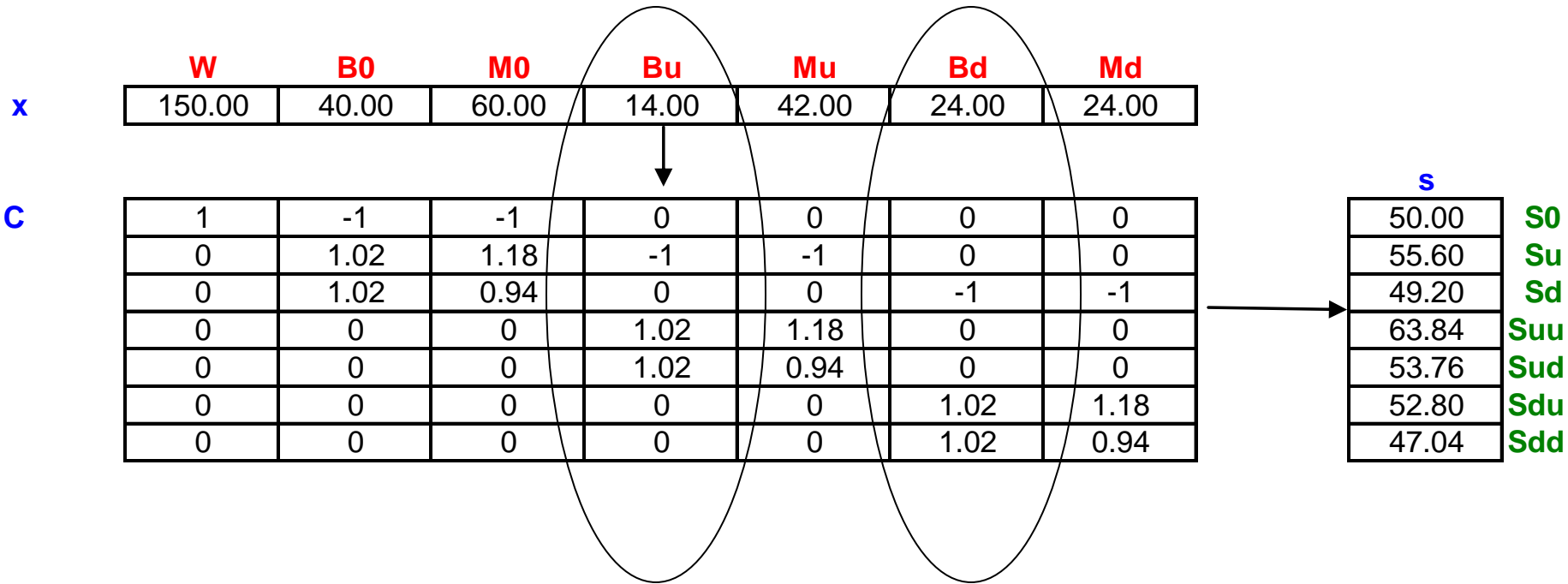
# Desired Spending: Multiple Periods



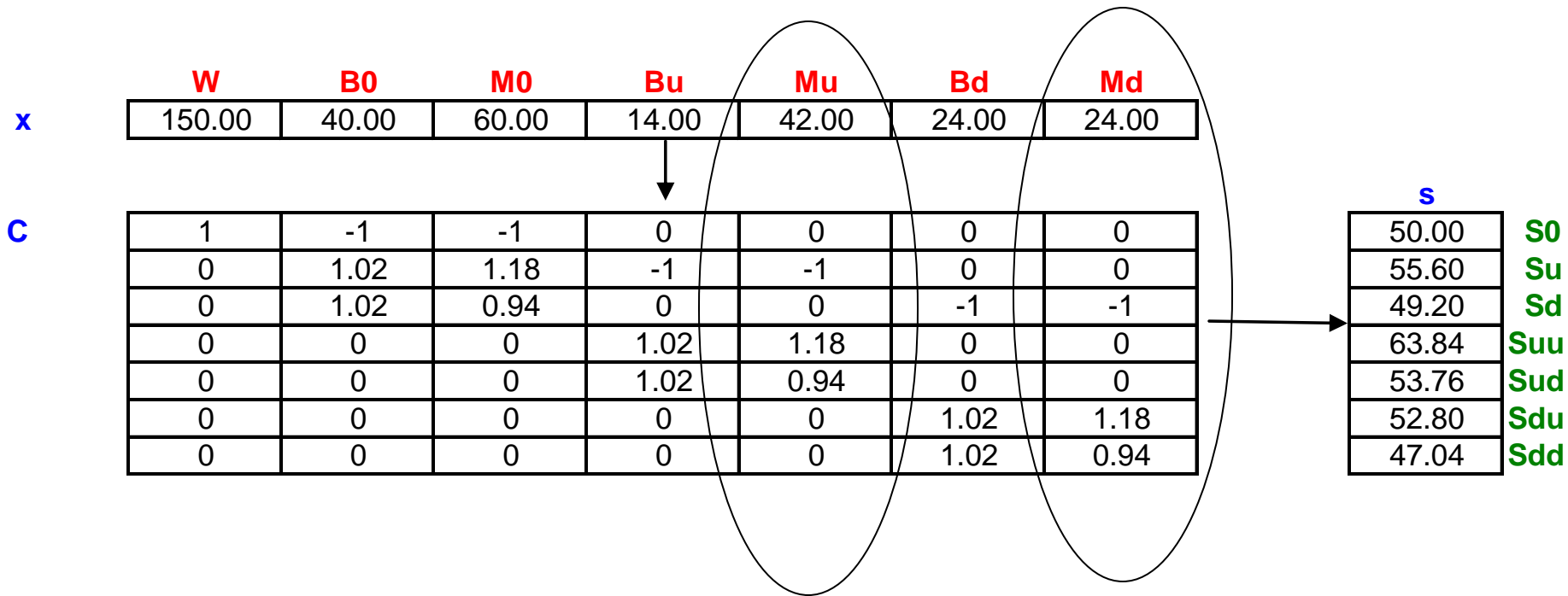
# Dynamic Strategies

|          |          |           |           |           |           |           |           |          |            |
|----------|----------|-----------|-----------|-----------|-----------|-----------|-----------|----------|------------|
|          | <b>W</b> | <b>B0</b> | <b>M0</b> | <b>Bu</b> | <b>Mu</b> | <b>Bd</b> | <b>Md</b> |          |            |
| <b>x</b> | 150.00   | 40.00     | 60.00     | 14.00     | 42.00     | 24.00     | 24.00     |          |            |
|          |          |           |           | ↓         |           |           |           |          |            |
| <b>C</b> | 1        | -1        | -1        | 0         | 0         | 0         | 0         |          |            |
|          | 0        | 1.02      | 1.18      | -1        | -1        | 0         | 0         |          |            |
|          | 0        | 1.02      | 0.94      | 0         | 0         | -1        | -1        |          |            |
|          | 0        | 0         | 0         | 1.02      | 1.18      | 0         | 0         |          |            |
|          | 0        | 0         | 0         | 1.02      | 0.94      | 0         | 0         |          |            |
|          | 0        | 0         | 0         | 0         | 0         | 1.02      | 1.18      |          |            |
|          | 0        | 0         | 0         | 0         | 0         | 1.02      | 0.94      |          |            |
|          |          |           |           |           |           |           |           | →        |            |
|          |          |           |           |           |           |           |           | <b>s</b> |            |
|          |          |           |           |           |           |           |           | 50.00    | <b>S0</b>  |
|          |          |           |           |           |           |           |           | 55.60    | <b>Su</b>  |
|          |          |           |           |           |           |           |           | 49.20    | <b>Sd</b>  |
|          |          |           |           |           |           |           |           | 63.84    | <b>Suu</b> |
|          |          |           |           |           |           |           |           | 53.76    | <b>Sud</b> |
|          |          |           |           |           |           |           |           | 52.80    | <b>Sdu</b> |
|          |          |           |           |           |           |           |           | 47.04    | <b>Sdd</b> |

# Contingent Bond Purchases



# Contingent Market Portfolio Purchases



# Lockbox, Period 2

% Bonds

32.89%

25.00%

50.00%

x

|   | W     | B0    | M0    | Bu    | Mu    | Bd    | Md    |
|---|-------|-------|-------|-------|-------|-------|-------|
| x | 49.67 | 16.34 | 33.33 | 14.00 | 42.00 | 24.00 | 24.00 |



C<sup>-1</sup>

|       |        |        |        |        |        |        |
|-------|--------|--------|--------|--------|--------|--------|
| 1.000 | 0.327  | 0.654  | 0.107  | 0.214  | 0.214  | 0.427  |
| 0.000 | -3.840 | 4.820  | -1.255 | -2.510 | 1.575  | 3.150  |
| 0.000 | 4.167  | -4.167 | 1.362  | 2.723  | -1.362 | -2.723 |
| 0.000 | 0.000  | 0.000  | -3.840 | 4.820  | 0.000  | 0.000  |
| 0.000 | 0.000  | 0.000  | 4.167  | -4.167 | 0.000  | 0.000  |
| 0.000 | 0.000  | 0.000  | 0.000  | 0.000  | -3.840 | 4.820  |
| 0.000 | 0.000  | 0.000  | 0.000  | 0.000  | 4.167  | -4.167 |

s

|       |     |
|-------|-----|
| 0.00  | S0  |
| 0.00  | Su  |
| 0.00  | Sd  |
| 63.84 | Suu |
| 53.76 | Sud |
| 52.80 | Sdu |
| 47.04 | Sdd |



# Lockbox Separation

- A retirement financial strategy is **fully specified** if spending in each year can be determined for any scenario of market returns
- A market is **complete** if any desired spending plan can be implemented with a retirement financial strategy
- If the market is complete, any fully specified retirement financial strategy can be implemented with a lockbox strategy

# Time-separable Multi-period Utility Functions

$$EU = EU(s_0) + EU(s_u, s_d) + EU(s_{uu}, s_{ud}, s_{du}, s_{dd})$$

$$EU(s_0) = u_0(s_0)$$

$$EU(s_u, s_d) = \pi_u u_1(s_u) + \pi_d u_1(s_d)$$

$$EU(s_{uu}, s_{ud}, s_{du}, s_{dd}) = \pi_{uu} u_2(s_{uu}) + \pi_{ud} u_2(s_{ud}) + \pi_{du} u_2(s_{du}) + \pi_{dd} u_2(s_{dd})$$

# Path-dependent Multi-period Utility Functions

$$p1 : s_0, s_u, s_{uu}$$

$$p2 : s_0, s_u, s_{ud}$$

$$p3 : s_0, s_d, s_{du}$$

$$p4 : s_0, s_d, s_{dd}$$

$$EU = \pi_{p1} u(p1) + \pi_{p2} u(p2) + \pi_{p3} u(p3) + \pi_{p4} u(p4)$$

# A Habit Formation Utility Function

$$u(C_t) = a_t (X - dC_{t-1})^{1-g} / (1-g)$$

# Issac Gonzales Survey, 2009

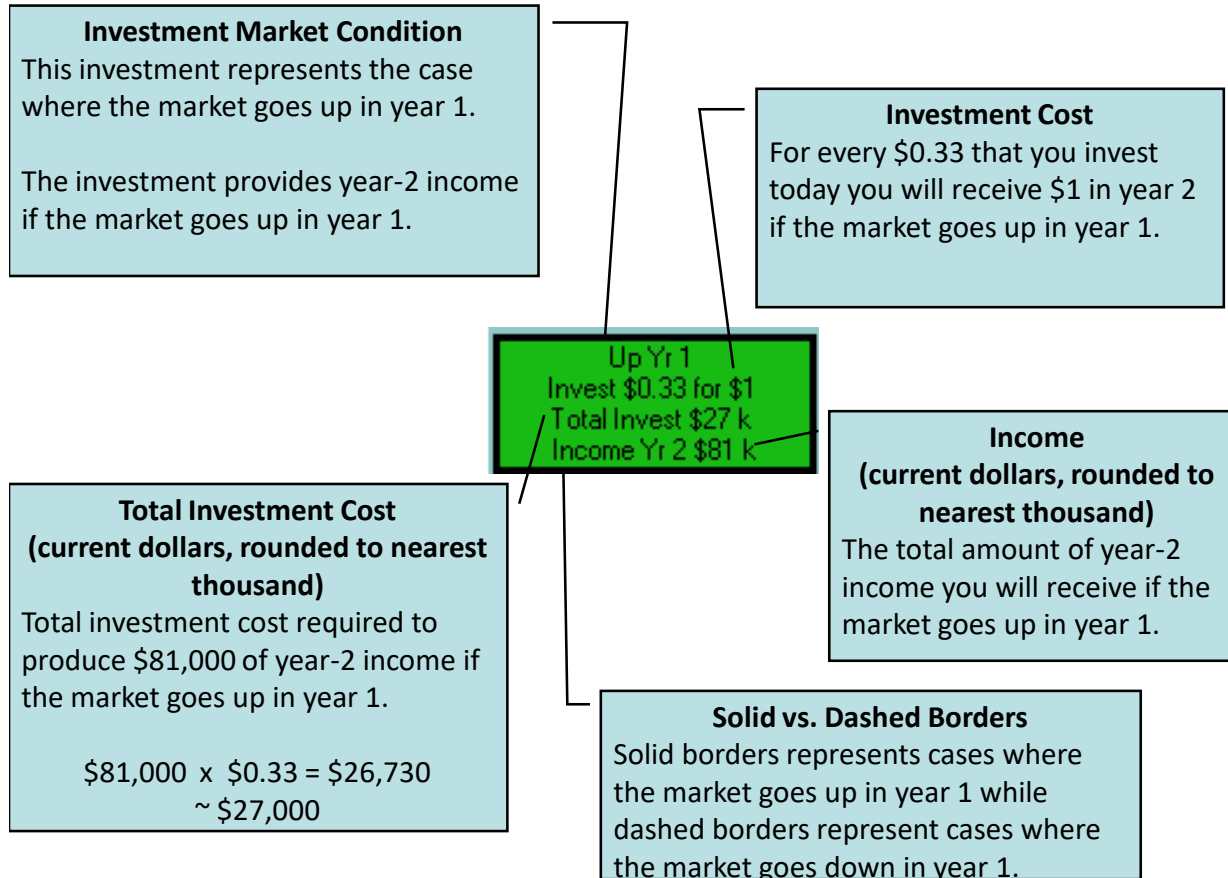
\$195 k remaining, continue to invest

[Click here to invest remainder of budget](#)

[Submit Survey](#)

| Retirement Year One                            | Retirement Year Two   | Retirement Year Three (market up previous year)   | Retirement Year Three (market down previous year)   |
|--|---|---|---|
|  |   |   |   |
| <p>Year 1 Income<br/>\$1 for \$1<br/>\$0 k</p> | <p>Up Yr 1<br/>Invest \$0.33 for \$1<br/>Total Invest \$0 k<br/>Income Yr 2 \$0 k</p> <p style="border: 1px dashed black;">Dn Yr 1<br/>Invest \$0.65 for \$1<br/>Total Invest \$0 k<br/>Income Yr 2 \$0 k</p> | <p>Up Yr 1, Up Yr 2<br/>Invest \$0.11 for \$1<br/>Total Invest \$0 k<br/>Income Yr 3 \$0 k</p> <p style="border: 1px dashed black;">Up Yr 1, Dn Yr 2<br/>Invest \$0.21 for \$1<br/>Total Invest \$0 k<br/>Income Yr 3 \$0 k</p> | <p style="border: 1px dashed black;">Dn Yr 1, Up Yr 2<br/>Invest \$0.21 for \$1<br/>Total Invest \$0 k<br/>Income Yr 3 \$0 k</p> <p style="border: 1px dashed black;">Dn Yr 1, Dn Yr 2<br/>Invest \$0.43 for \$1<br/>Total Invest \$0 k<br/>Income Yr 3 \$0 k</p> |

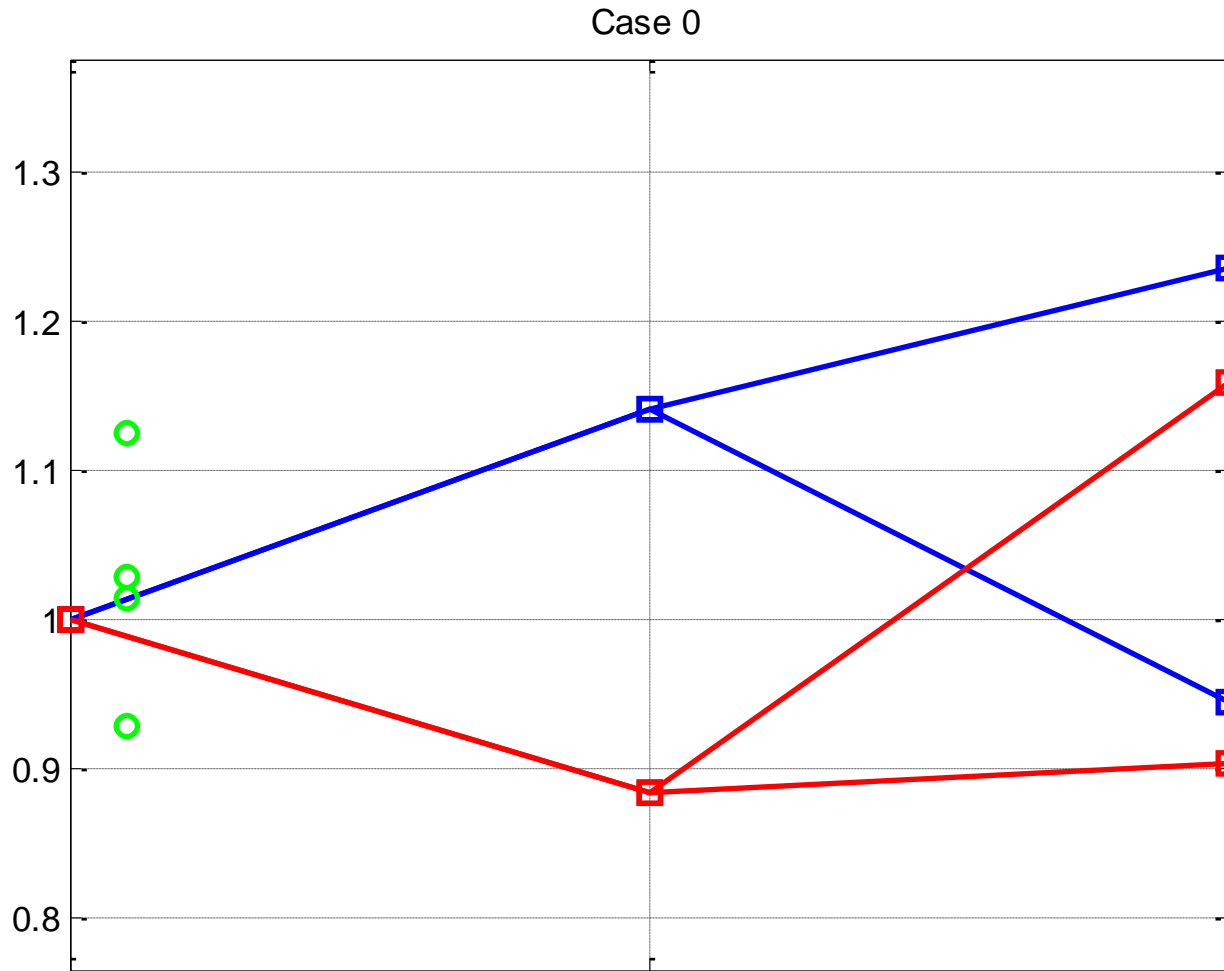
# Survey Details



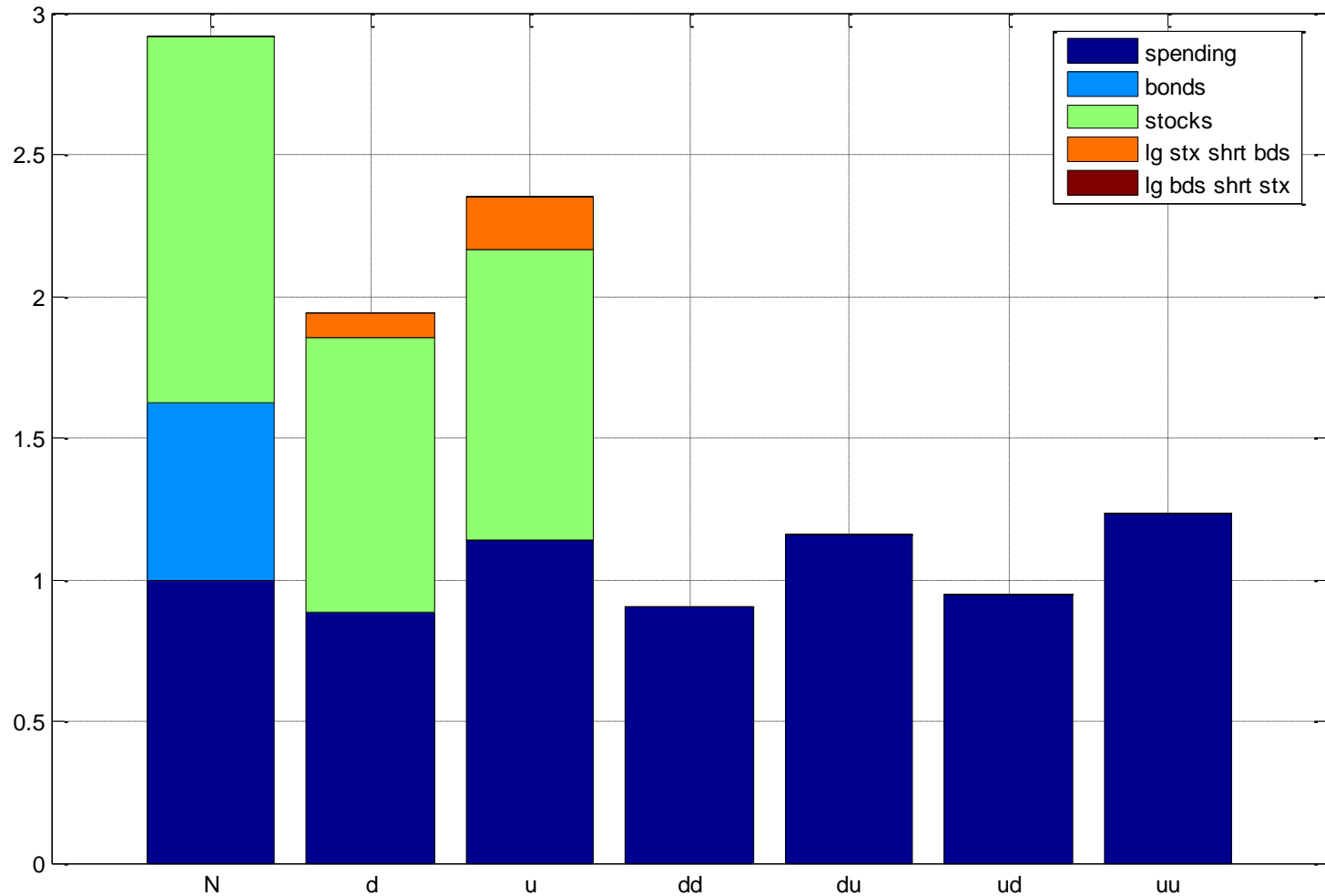
# Survey Example

| <a href="#">\$0 remaining, submit or readjust if desired</a> |   | <a href="#">Click here to invest remainder of budget</a>   |   | <a href="#">Submit Survey</a> |
|--|---|--|---|-------------------------------|
| Retirement Year One  | Retirement Year Two   | Retirement Year Three (market up previous year)  | Retirement Year Three (market down previous year)   |                               |
| <p>Year 1 Income<br/>\$1 for \$1</p> <p>\$65 k</p>           | <p>Up Yr 1<br/>Invest \$0.33 for \$1<br/>Total Invest \$25 k<br/>Income Yr 2 \$77 k</p> <p>Dn Yr 1<br/>Invest \$0.65 for \$1<br/>Total Invest \$40 k<br/>Income Yr 2 \$61 k</p> | <p>Up Yr 1, Up Yr 2<br/>Invest \$0.11 for \$1<br/>Total Invest \$9 k<br/>Income Yr 3 \$82 k</p> <p>Up Yr 1, Dn Yr 2<br/>Invest \$0.21 for \$1<br/>Total Invest \$13 k<br/>Income Yr 3 \$63 k</p> | <p>Dn Yr 1, Up Yr 2<br/>Invest \$0.21 for \$1<br/>Total Invest \$17 k<br/>Income Yr 3 \$81 k</p> <p>Dn Yr 1, Dn Yr 2<br/>Invest \$0.43 for \$1<br/>Total Invest \$26 k<br/>Income Yr 3 \$61 k</p> |                               |

# Average Response

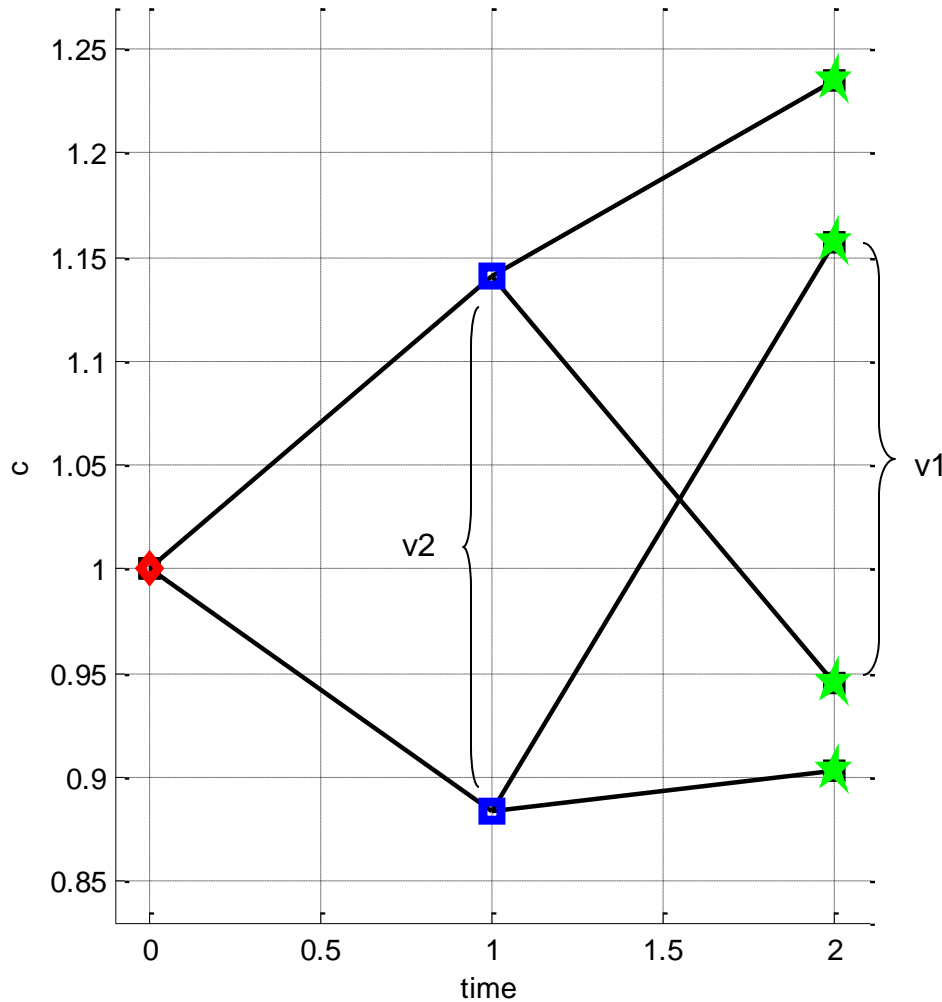


# Required Financial Strategy



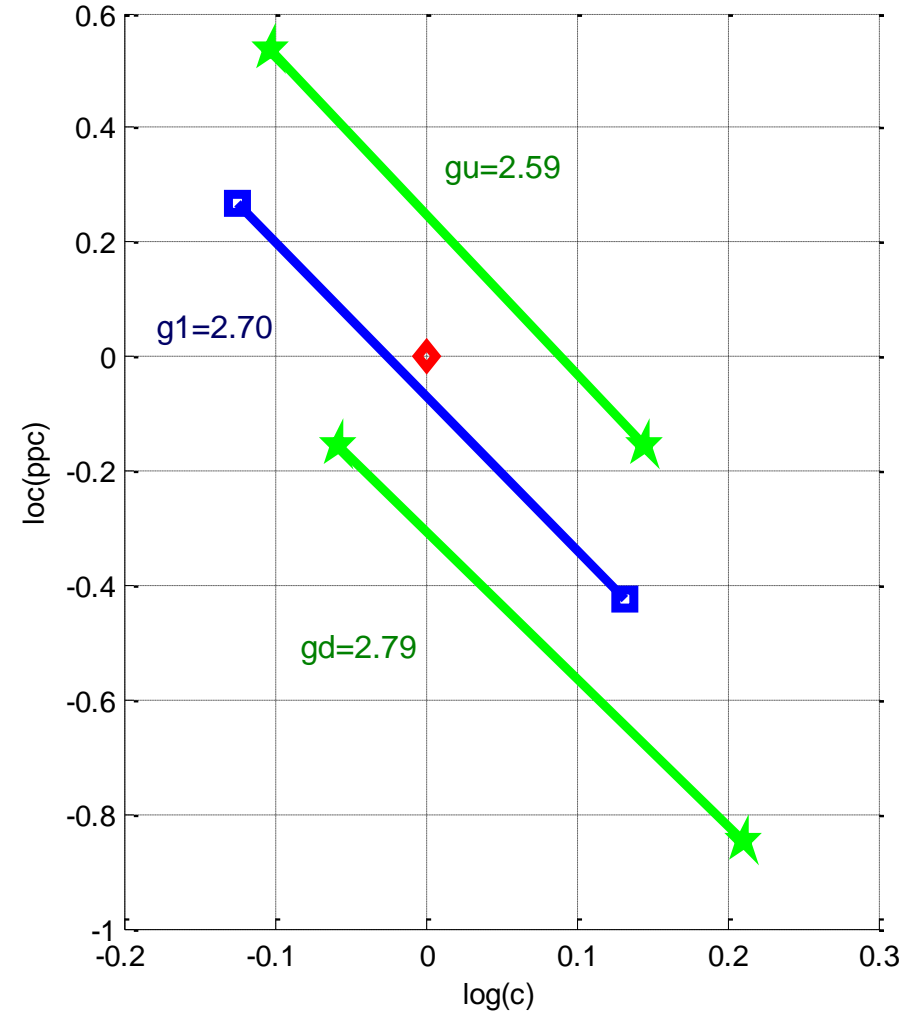
# Implied Marginal Utility Functions

Consumption versus time

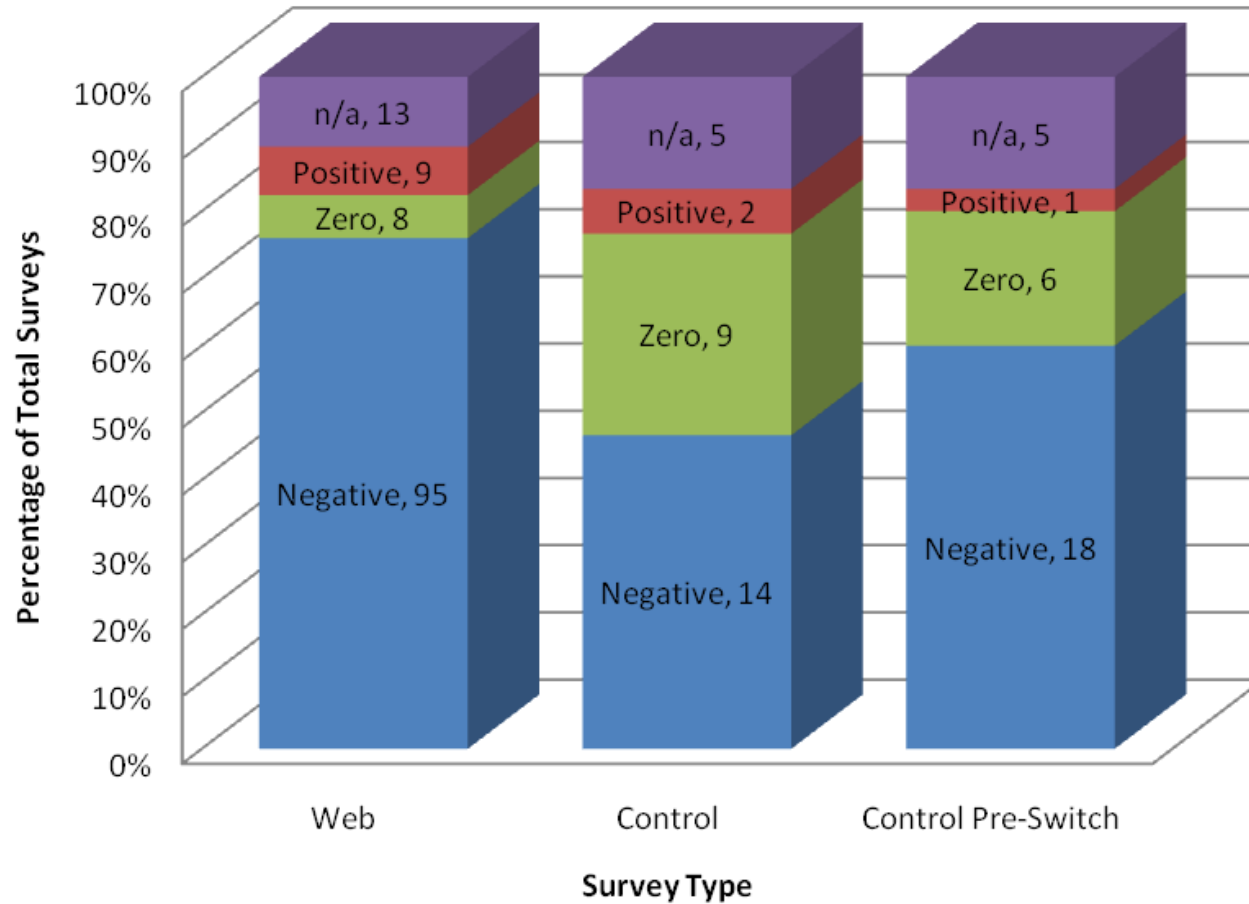


$$d = v1/v2 = -0.82$$

log(PPC) versus log(c)



# d Values



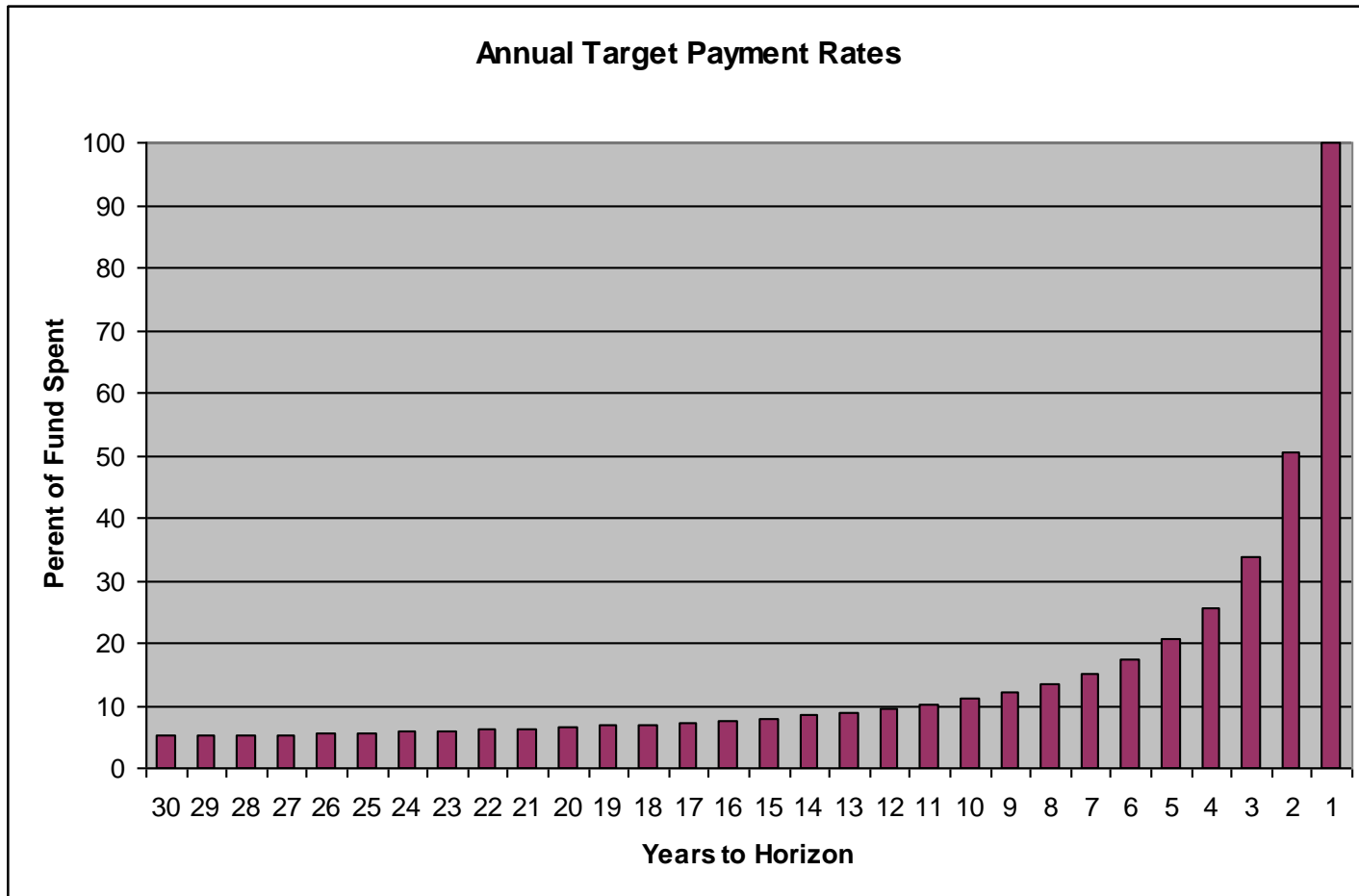
# Unanswered Questions

- How can we determine an individual's true preferences?
- Are individual choices consistent with axioms of “rational decisions”?
- How can the influence of framing be minimized?
- How can an optimal financial strategy for complex preferences be determined?

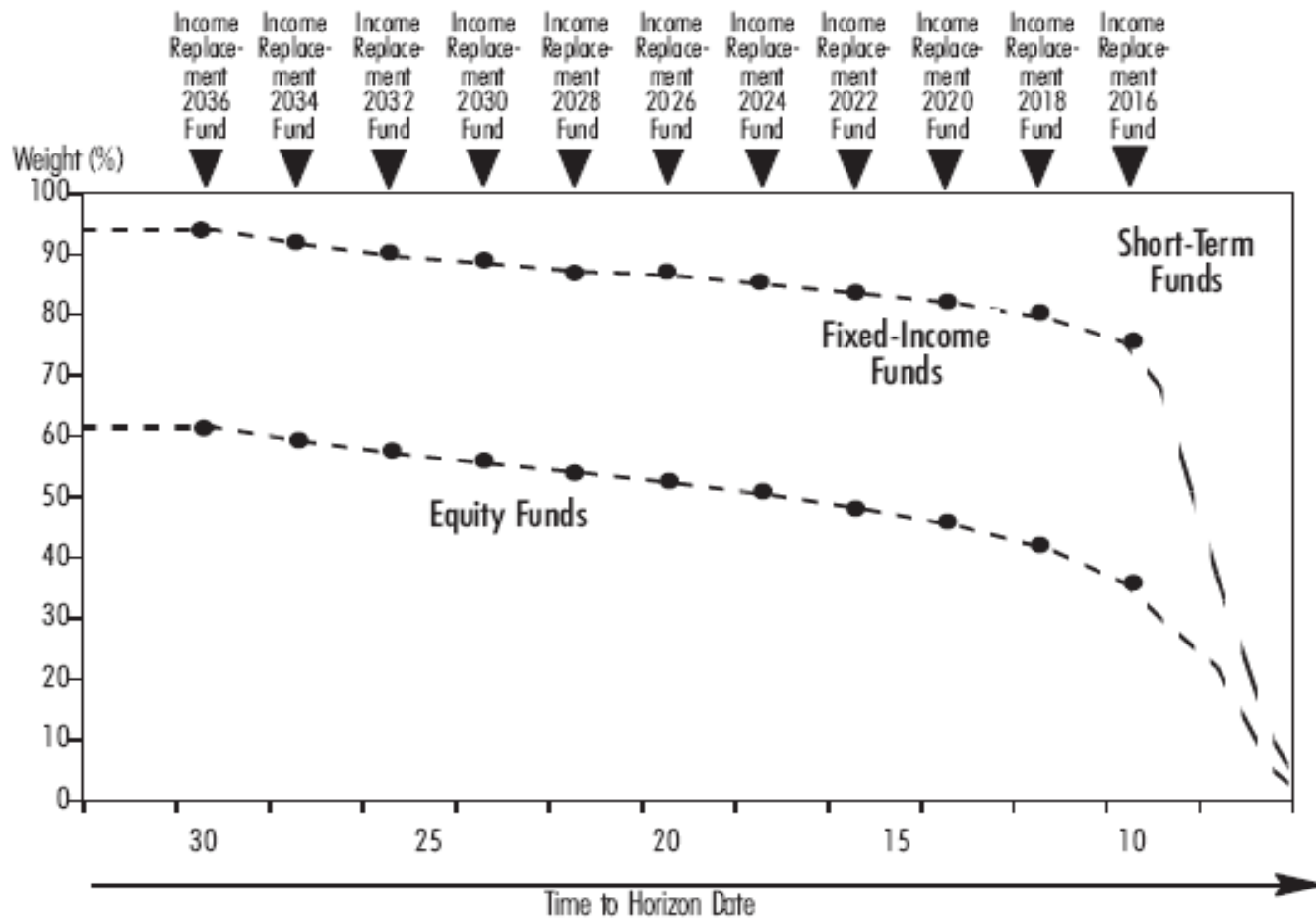
# The Fidelity Income Replacement Funds

- Horizon date
  - E.g. 2036
- Investment strategy
  - Time-dependent “glide path” asset allocation
- Spending Rule
  - Pre-specified time-dependent proportions of asset value

# Spending Rule



# Investment Strategy



# Lockbox Equivalence

- Any strategy with a time-dependent proportional spending rule and a time-dependent investment strategy is equivalent to a lockbox strategy
- Each lockbox will have the same investment strategy and
- The initial amounts to be invested in the lockboxes can be computed from the pre-specified spending rates

# Initial Lockbox Values (1)

- Let:
  - $k_t$  = the proportion spent in year  $t$
  - $R_t$  = the total return on investment in year  $t$   
(e.g. 1.02 for 2%)
- The amounts spent in the first three years will be:
  - $Wk_0$
  - $(1-k_0)WR_1k_1$
  - $(1-k_0)WR_1(1-k_1)R_2k_2$

# Initial Lockbox Values (2)

- Re-arranging:

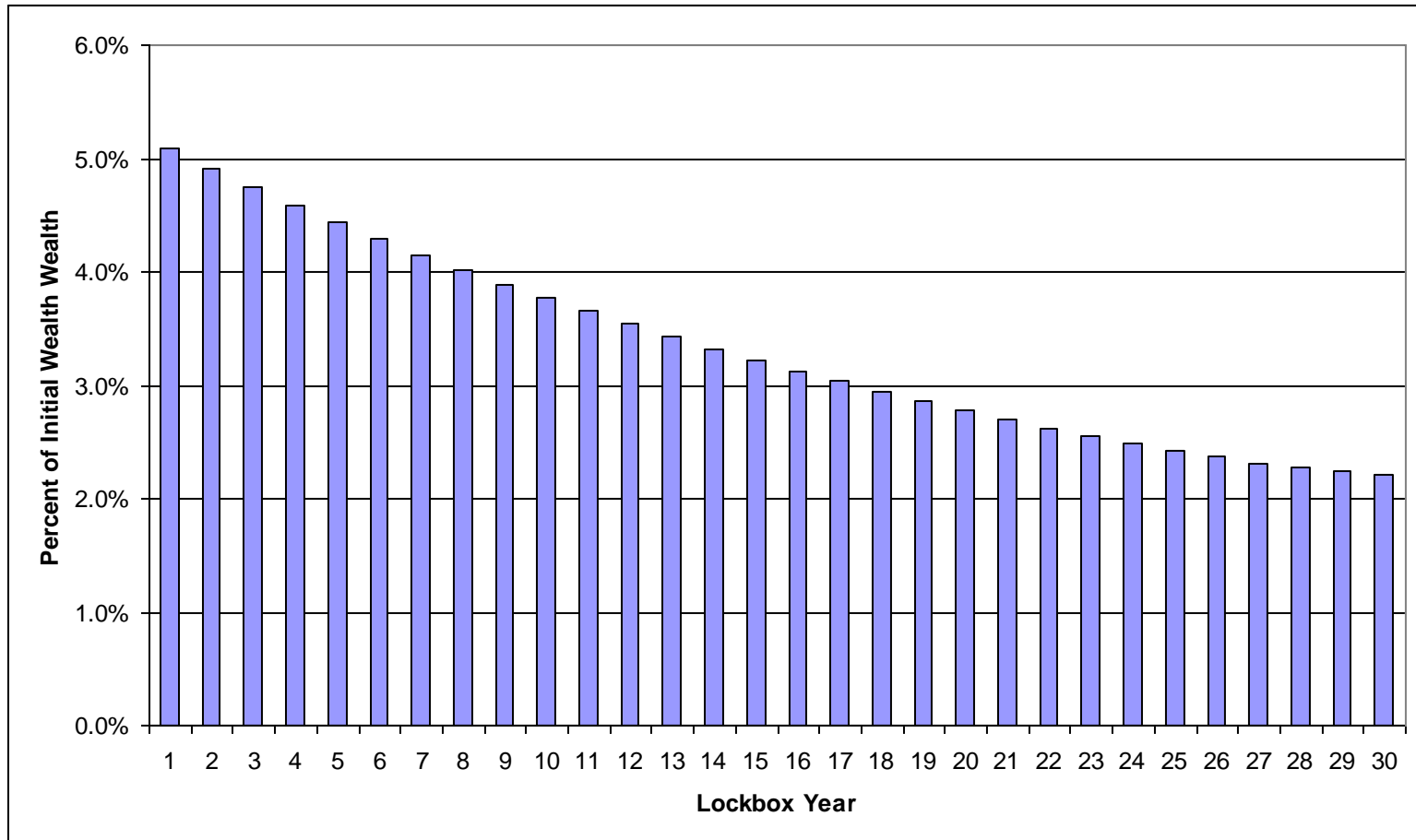
$$\{Wk_0\}$$

$$\{W(1-k_0)k_1\} R_1$$

$$\{W(1-k_0)(1-k_1)k_2\} R_1R_2$$

- But these are the ending values for lockboxes with the initial investments shown in the brackets { }
  - therefore, investing these amounts in lockboxes will give the same spending plan as the original strategy

# Percentages of Initial Wealth in Lockboxes



# A Simple Income Replacement Fund

- Two assets
  - A riskless real bond
  - A market portfolio
    - (e.g. 60% Stocks, 40% Bonds)
- A glide path similar to that for equity funds in the Fidelity Income Replacement Funds
- A 30-year horizon
- Annual payment rates equal to those of the Fidelity Income Replacement Funds

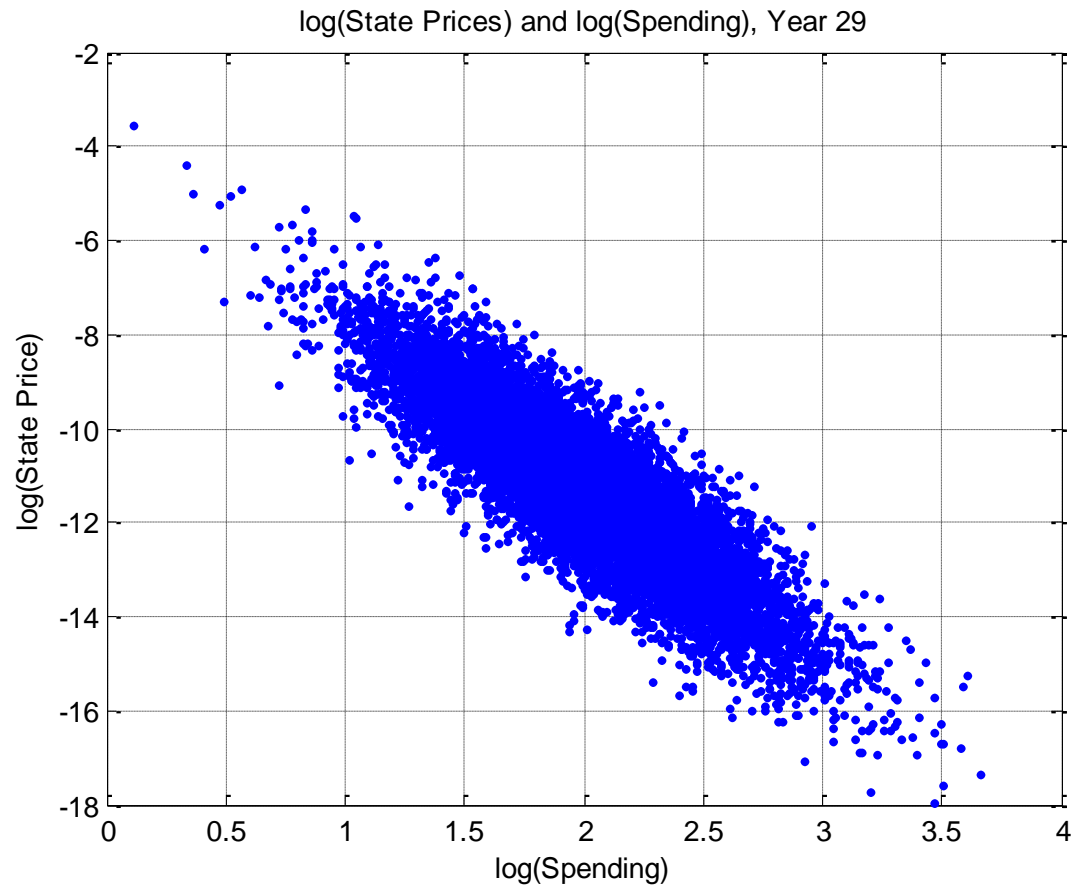
# Capital Market Characteristics

- Riskless real return
  - 2 % per year
- Market portfolio real return
  - Lognormally distributed each year
  - Expected annual return
    - 6 % per year
  - Annual standard deviation of return
    - 12 % per year
  - No serial correlation

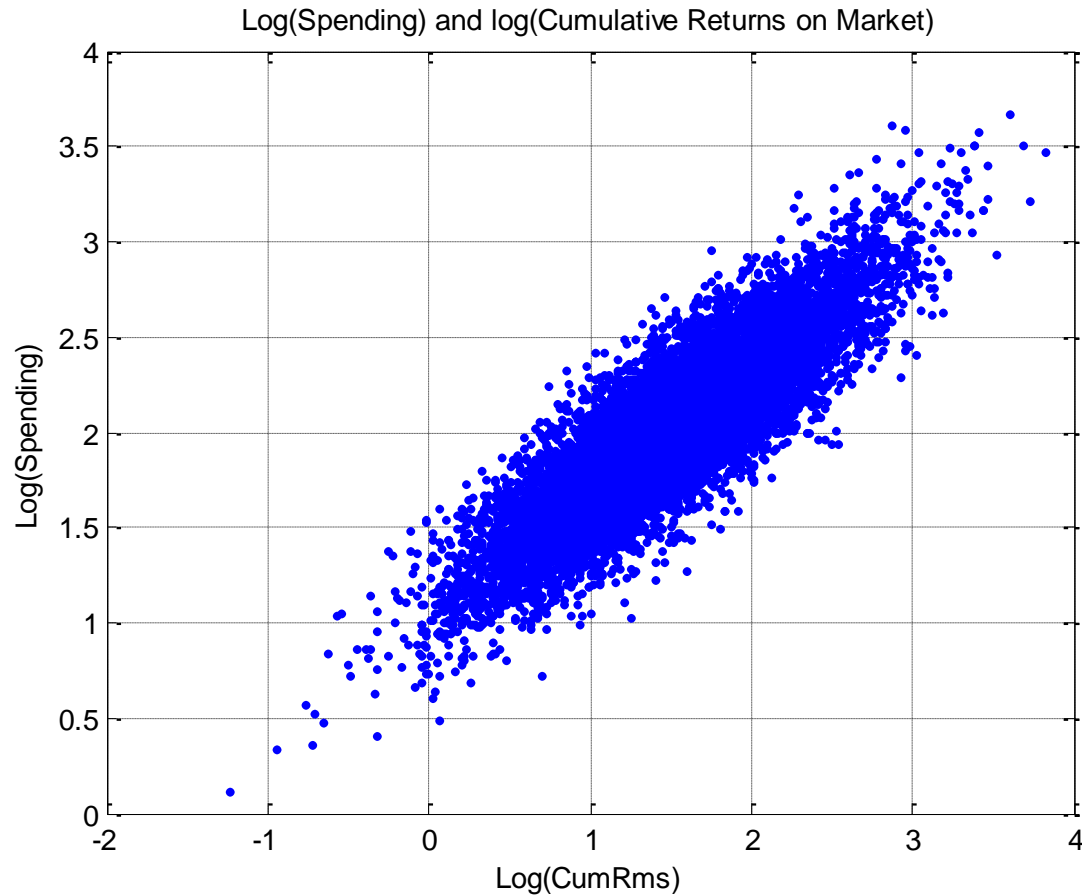
# Monte Carlo Simulations

- 10,000 scenarios of 29 years each
- Returns for each lockbox are simulated
- State prices for payment in year 29 are assumed to be log-linearly related to cumulative market returns
  - Consistent with a CRRA pricing kernel
  - Consistent with limit of a binomial i.i.d. return-generating process

# Year 29: State Prices and Spending



# Year 29: Cumulative Market Return and Spending



# Real-world Challenges

- Determining each individual's true preferences
- Determining the return generating process
- Representing capital market instruments
- Estimating the feasibility of dynamic strategies
- Incorporating annuities
- Insuring the macro-consistency of optimal strategies